Cournot-Walras Equilibrium as A Subgame Perfect Equilibrium*

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Abstract

In this paper, we respecify à la Cournot-Walras the mixed version of a model of noncooperative exchange, originally proposed by Lloyd S. Shapley. We first show that this respecification has a Cournot-Walras equilibrium allocation, which does not correspond to any Cournot-Nash equilibrium of the mixed version of the original Shapley's model. As this is due to the intrinsic two-stage nature of the Cournot-Walras equilibrium concept, we are led to consider a further reformulation of the Shapley's model as a two-stage game, where the atoms move in the first stage and the atomless sector moves in the second stage. Our main result shows that any Cournot-Walras equilibrium allocation corresponds to a subgame perfect equilibrium of this two-stage game, thereby providing a game theoretical foundation to the Cournot-Walras equilibrium approach. *Journal of Economic Literature* Classification Numbers: C72, D51.

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1 Introduction

One of the first attempts to extend the analysis of oligopolistic interaction proposed by Cournot to a general equilibrium framework was due to Gabszewicz and Vial (1972), who introduced the concept of Cournot-Walras equilibrium. They considered an economy with production where firms are assumed to be "few" whereas consumers are assumed to be "many." Firms produce consumption goods and distribute them, according to some preassigned shares, to consumers, who are therefore endowed with their initial endowments plus the bundles of consumption goods which they receive as shareholders of the firms. Consumers are then allowed to exchange their endowments among themselves and the equilibrium prices resulting from these exchanges enable firms to determine the profits associated with their production decisions. A Cournot-Walras equilibrium is a noncooperative equilibrium of a game where the players are the firms, the strategies are their production decisions and the payoffs are their profits. The denomination of this equilibrium concept comes from the fact that firms behave "à la Cournot" in making their production decisions while consumers behave "à la Walras" in exchanging goods. The line of research initiated by Gabszewicz and Vial (1972) raised some theoretical problems (see also Roberts and Sonnenschein (1977), Roberts (1980), Mas-Colell (1982), Dierker and Grodal (1986), among others). Gabszewicz and Vial (1972) were already aware that their concept of Cournot-Walras equilibrium depends on the rule chosen to normalize prices and that profit maximization may not be a rational objective for the firms.

In order to overcome these problems, Codognato and Gabszewicz (1991) introduced a Cournot-Walras equilibrium concept for exchange economies where "few" traders, called the oligopolists, behave strategically "à la Cournot" in making their supply decisions and share the endowment of a particular commodity while "many small" traders behave "à la Walras" and share the endowments of all the other commodities. The oligopolists are allowed to supply a fraction of their initial endowments. Taking prices as given, each oligopolist is able to determine the income corresponding to his supply decision and to choose the bundle of commodities which gives him the highest utility. All traders, behaving "à la Walras," are then allowed to exchange commodities among themselves until prices clear all the markets. A Cournot-Walras equilibrium is a noncooperative equilibrium of a game where the players are the oligopolists, the strategies are their supply decisions and the payoffs are the utility levels they achieve through the exchange. The line of research initiated by Codognato and Gabszewicz (1991) circumvented the theoretical difficulties mentioned above by defining an equilibrium concept which does not depend on price normalization and by replacing profit maximization with utility maximization (see also Codognato and Gabszewicz (1993), Codognato (1995), d'Aspremont et al. (1997), Gabszewicz and Michel (1997), Shitovitz (1997), Lahmandi-Ayed (2001), among others). Nevertheless, the whole Cournot-Walras equilibrium approach is not immune from another fundamental criticism. In fact, all the models mentioned above do not explain why a particular agent has a strategic behaviour or a competitive one.

Taking inspiration from the cooperative approach to oligopoly introduced by Shitovitz (1973), Okuno et al. (1980) proposed a foundation of agents' behaviour that considered the Cournot-Nash equilibria of a model of simultaneous, noncooperative exchange between large traders, represented as atoms, and small traders, represented by an atomless sector. The model of noncooperative exchange they used belongs to a line of research initiated by Shapley and Shubik (1977) (see also Dubey and Shubik (1977), Postlewaite and Schmeidler (1978), Mas-Colell (1982), Amir et al. (1990), Peck et al. (1992), Dubey and Shapley (1994), among others). In particular, Okuno et al. (1980) showed that large traders keep their strategic power even when their behaviour turns out to be competitive in the cooperative framework considered by Shitovitz (1973). Codognato and Ghosal (2000b) generalized the analysis of Okuno et al. (1980) by considering a mixed version of a model of noncooperative exchange, originally proposed by Lloyd S. Shapley, and analysed by Sahi and Yao (1989) for exchange economies with a finite number of traders, and by Codognato and Ghosal (2000a) for exchange economies with an atomless continuum of traders. In this model, traders send out bids, i.e., quantity signals, which indicate how much of each commodity they are willing to offer for trade. Every bid of each commodity is tagged by the name of some other commodity for which it has to be exchanged. The rule of price formation requires that a single price system, which equates the value of the total amount of bids of any commodity to the value of the total amount available of that commodity, is used to clear the markets. By comparing the mixed version of the Shapley's model with the mixed version of the model in Codognato and Gabszewicz (1991), Codognato (1995) already showed, through an example, that there is a Cournot-Walras equilibrium allocation which does not correspond to any Cournot-Nash equilibrium. There could be two reasons for this result. The first is that the Cournot-Walras equilibrium concept has an intrinsic two-stage nature which cannot be reconciled with the one-stage Cournot-Nash equilibrium of the Shapley's model. The second is that, in the model by Codognato and Gabszewicz (1991), the oligopolists behave à la Cournot in making their supply decisions and à la Walras in exchanging commodities whereas, in the mixed version of the Shapley's model, the large traders behave unambigously à la Cournot. This "twofold behaviour" of large traders raises a further problem in the line of research introduced by Codognato and Gabszewicz (1991) which should be dealt with.

To this end, here we introduce a respecification à la Cournot-Walras of the mixed version of the Shapley's model. In particular, we assume that large traders behave à la Cournot in making bids, as in the Shapley's model, while the atomless sector is assumed to behave à la Walras. Given the atoms' bids, prices adjust to equate the aggregate net bids to the aggregate net demands of the atomless sector. Each nonatomic trader then obtains his Walrasian demand whereas each large trader obtains final holdings determined as in the Shapley's model. A Cournot-Walras equilibrium is a noncooperative equilibrium of a game where the players are the large traders, the strategies are their bids and the payoffs are the utility levels they achieve through the exchange process described above. We provide an example which shows that there is an allocation corresponding to the Cournot-Walras equilibrium just defined which does not correspond to the Cournot-Walras equilibrium defined by Codognato and Gabszewicz (1991). We also show, through an example, that there is an allocation corresponding to the Cournot-Walras equilibrium of our variant of the mixed version of the Shapley's model which does not correspond to any Cournot-Nash equilibrium of the mixed version of the original Shapley's model. This confirms, within a different framework, the result obtained by Codognato (1995). Since now large traders behave unambigously à la Cournot in both models, we could guess that this result is explained by the two-stage implicit nature of the Cournot-Walras equilibrium concept. Therefore, we consider a two-stage game which is a reformulation of the mixed version of the Shapley's model where the atoms move in the first stage and the atomless sector move in the second stage. Our main result then follows. Any Cournot-Walras equilibrium allocation corresponds to a subgame perfect equilibrium of the game sketched above. This theorem reconciles the line of research initiated by Shapley and Shubik (1977) with the Cournot-Walras approach and makes this approach immune from the criticism by Okuno et al. (1980), as it provides a game-theoretical foundation of strategic and competitive behaviour.

The paper is organized as follows. In Section 2, we define our Cournot-Walras equilibrium concept. In Section 3, we compare this equilibrium concept with other equilibrium concepts for exchange economies. In Section 4, we show our main theorem which states that our Cournot-Walras equilibrium is a subgame perfect equilibrium. Section 5 concludes.

2 The model

We are working in the Euclidean space R^l_+. The dimension l represents the number of different commodities being traded in the market. We denote by $(x^1, \dots, x^l)$ a vector of R^l_+. Traders are represented by the elements of the set $T = T_0 \cup T_1$, where $T_0 = [0,1]$ and $T_1 = \{2, \dots, m+1\}$. Following Codognato and Ghosal (2000b), it is possible to denote the space of traders by the complete measure space $(T, \mathcal{T}, \mu)$, where $\mathcal{T}$ denotes the σ-algebra of all μ-measurable subsets of T and μ is a measure which is the Lebesgue measure, when restricted to $\mathcal{T}_{T_0} = \{D \cap T_0 : D \in \mathcal{T}\}$, and the counting measure, when restricted to $\mathcal{T}_{T_1} = \{D \cap T_1 : D \in \mathcal{T}\}$. By Propositions 3 and 4 in Codognato and Ghosal (2000b), it is straightforward to show that the measure space $(T_0, \mathcal{T}_{T_0}, \mu)$ is atomless and the measure space $(T_1, \mathcal{T}_{T_1}, \mu)$ is purely atomic; moreover, for each $t \in T_1$, the singleton set $\{t\}$ is an atom of the measure space $(T, \mathcal{T}, \mu)$ (see, for instance, Aliprantis and Border (1999), p. 357). A null set of traders is a set of Lebesgue measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "all" traders, or "each" trader, or "each" trader in a certain set, is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue. Given any function $\mathbf{f}$ defined on T, we denote by ${}^0\mathbf{f}$ and ${}^1\mathbf{f}$ the restrictions of $\mathbf{f}$ to T_0 and T_1, respectively. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumptions.

Assumption 1. $\mathbf{w}(t) > 0$, *for all* $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by an utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^l_+ \to R$ is continuous, strictly monotonic, strictly quasi-concave, for all $t \in T$.*

Assumption 3. *$u : T \times R^l_+ \to R$, given by $u(t, x) = u_t(x)$, is measurable.*

A price vector is a vector $p \in R^l_+$. According to Aumann (1966), we define, for each $p \in R^l_+$, a correspondence $\boldsymbol{\Delta}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\boldsymbol{\Delta}_p(t) = \{x \in R^l_+ : px \leq p\mathbf{w}(t)\}$, and a correspondence $\boldsymbol{\Gamma}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\boldsymbol{\Gamma}_p(t) = \{x \in R^l_+ : \text{ for all } y \in \boldsymbol{\Delta}_p(t), u_t(x) \geq u_t(y)\}$. A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that, for all $t \in T$, $\mathbf{x}^*(t) \in \boldsymbol{\Delta}_{p^*}(t) \cap \boldsymbol{\Gamma}_{p^*}(t)$. We proceed now to formulate the concept of Cournot-Walras equilibrium. By Assumption 2, it is possible to define, for each $p \in R^l_{++}$, a function ${}^0\mathbf{x}(\cdot, p) : T_0 \to R^l_+$, such that, for each $t \in T_0$, ${}^0\mathbf{x}(t, p) = \boldsymbol{\Delta}_p(t) \cap \boldsymbol{\Gamma}_p(t)$. We are now able to show the following proposition.

Proposition 1. *For each $p \in R^l_{++}$, the function ${}^0\mathbf{x}(\cdot, p)$ is integrable.*

Proof. Let $p \in R^l_{++}$. From Aumann (1966), we know that the function ${}^0\mathbf{x}(\cdot, p)$ is a Borel measurable function since the correspondences $\boldsymbol{\Delta}_p$ and $\boldsymbol{\Gamma}_p$ are Borel measurable and ${}^0\mathbf{x}(t, p) = \boldsymbol{\Delta}_p(t) \cap \boldsymbol{\Gamma}_p(t)$, for each $t \in T_0$. Moreover, ${}^0\mathbf{x}(\cdot, p)$ is integrably bounded since ${}^0\mathbf{x}^i(t, p) \leq \frac{\sum_{j=1}^l p^j \mathbf{w}^j(t)}{p^i}$, $i = 1, \ldots, l$, for all $t \in T_0$. But then, by Theorem 2 in Aumann (1965), the function ${}^0\mathbf{x}(\cdot, p)$ is integrable. ■

Let $e \in R^{l^2}$ be a vector such that $e = (e_{11}, e_{12}, \ldots, e_{ll-1}, e_{ll})$. A strategy correspondence is a correspondence $\mathbf{E} : T_1 \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T_1$, $\mathbf{E}(t) = \{e \in R^{l^2} : e_{ij} \geq 0,\, i, j = 1, \ldots, l;\, \sum_{j=1}^l e_{ij} \leq \mathbf{w}^i(t),\, i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{e} : T_1 \to R^{l^2}$ such that, for all $t \in T_1$, $\mathbf{e}(t) \in \mathbf{E}(t)$. For each $t \in T_1$, $\mathbf{e}_{ij}(t)$, $i, j = 1, \ldots, l$, is the amount of commodity i that trader t offers in exchange for commodity j. Let E be the set of all strategy selections. Moreover, let $\mathbf{e} \setminus e(t)$ be a strategy selection obtained by replacing $\mathbf{e}(t)$ in $\mathbf{e}$ with $e(t) \in \mathbf{E}(t)$. Finally, let $\pi(\mathbf{e})$ denote the correspondence which associates, to each $\mathbf{e} \in E$, the set of the price vectors such that

$$\int_{T_0} {}^0\mathbf{x}^j(t,p)\,d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}_{ij}(t)\,d\mu \frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\,d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}_{ji}(t)\,d\mu, \qquad (1)$$

$j = 1, \ldots, l$.

Assumption 4. *For each* $\mathbf{e} \in E$, $\pi(\mathbf{e}) \neq \emptyset$ *and* $\pi(\mathbf{e}) \subset R^l_{++}$.

A price selection $p(\mathbf{e})$ is a function which associates, to each $\mathbf{e} \in E$, a price vector $p \in \pi(\mathbf{e})$ and is such that $p(\mathbf{e}') = p(\mathbf{e}'')$ if $\int_{T_1} \mathbf{e}'(t)\,d\mu = \int_{T_1} \mathbf{e}''(t)\,d\mu$. For each strategy selection $\mathbf{e} \in E$, let ${}^1\mathbf{x}(\cdot, \mathbf{e}(\cdot), p(\mathbf{e})) : T_1 \to R^l_+$ denote a function such that

$${}^1\mathbf{x}^j(t, \mathbf{e}(t), p(\mathbf{e})) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{e}_{ji}(t) + \sum_{i=1}^{l} \mathbf{e}_{ij}(t) \frac{p^i(\mathbf{e})}{p^j(\mathbf{e})}, \qquad (2)$$

for all $t \in T_1$, $j = 1, \ldots, l$. Given a strategy selection $\mathbf{e} \in E$, taking into account the structure of the traders' measure space, Proposition 1, and Equation (1), it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = {}^0\mathbf{x}(t, p(\mathbf{e}))$, for all $t \in T_0$, and $\mathbf{x}(t) = {}^1\mathbf{x}(t, \mathbf{e}(t), p(\mathbf{e}))$, for all $t \in T_1$, is an allocation. At this stage, we are able to define the concept of Cournot-Walras equilibrium.

Definition 1. *A pair* $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, *consisting of a strategy selection* $\tilde{\mathbf{e}}$ *and an allocation* $\tilde{\mathbf{x}}$ *such that* $\tilde{\mathbf{x}}(t) = {}^0\mathbf{x}(t, p(\tilde{\mathbf{e}}))$, *for all* $t \in T_0$, *and* $\tilde{\mathbf{x}}(t) = {}^1\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))$, *for all* $t \in T_1$, *is a Cournot-Walras equilibrium, with respect to a price selection* $p(\mathbf{e})$, *if* $u_t({}^1\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))) \geq u_t({}^1\mathbf{x}(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t))))$, *for all* $t \in T_1$ *and for all* $e(t) \in \mathbf{E}(t)$.

3 Cournot-Walras, Walras, and Cournot-Nash equilibrium

In this Section, we compare the Cournot-Walras equilibrium concept introduced above with other equilibrium concepts for exchange economies. We first investigate the relationship between the concepts of Cournot-Walras and Walras equilibrium. As is well known, within the Cournotian tradition, it has been established that the Cournot equilibrium approaches the competitive equilibrium as the number of oligopolists increases. We confirm this result for our Cournot-Walras equilibrium, by directly considering a limit exchange economy, i.e., an exchange economy with an atomless continuum of oligopolists. More precisely, let the space of traders be denoted by the complete measure space $(T, \mathcal{T}, \mu)$, where the set of traders is denoted by $T = T_0 \cup T_1$, with $T_0 = [0, 1]$ and $T_1 = [2, 3]$, $\mathcal{T}$ is the σ-algebra of all measurable subsets of T, and μ is the Lebesgue measure on $\mathcal{T}$. The following Proposition shows that, in this framework, the set of the Cournot-Walras equilibrium allocations coincides with the set of the Walras equilibrium allocations.

Proposition 2. *Under Assumptions 1, 2, 3, and 4, (i) if $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot-Walras equilibrium with respect to a price selection $p(\mathbf{e})$, there is a price vector $\tilde{p}$ such that $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium; (ii) if $(p^*, \mathbf{x}^*)$ is a Walras equilibrium, there is a strategy selection $\mathbf{e}^*$ such that $(\mathbf{e}^*, \mathbf{x}^*)$ is a Cournot-Walras equilibrium with respect to a price selection $p(\mathbf{e})$.*

Proof. (i) Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot-Walras equilibrium with respect to the price selection $p(\mathbf{e})$. First, it is straightforward to show that, for all $t \in T_1$, $\tilde{p}\tilde{\mathbf{x}}(t) = \tilde{p}\mathbf{w}(t)$, where $\tilde{p} = p(\tilde{\mathbf{e}})$. Let us now show that, for all $t \in T_1$, $\tilde{\mathbf{x}}(t) \in \boldsymbol{\Delta}_{\tilde{p}}(t) \cap \boldsymbol{\Gamma}_{\tilde{p}}(t)$. Suppose that this is not the case for a trader $t \in T_1$. Then, by Assumption 2, there is a bundle $z \in \{x \in R^l_+ : \tilde{p}x = \tilde{p}\mathbf{w}(t)\}$ such that $u_t(z) > u_t(\tilde{\mathbf{x}}(t))$. By Lemma 5 in Codognato and Ghosal (2000a), there exist $\lambda^j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^j = 1$, such that

$$z^j = \lambda^j \frac{\sum_{j=1}^l \tilde{p}^j \mathbf{w}^j(t)}{\tilde{p}^j},\ j = 1, \ldots, l.$$

Let $e_{ij}(t) = \mathbf{w}^i(t)\lambda^j$, $i, j = 1, \ldots, l$. Substituting in Equation (2) and taking into account the fact that, by Equation (1), $p(\tilde{\mathbf{e}}) = p(\tilde{\mathbf{e}} \setminus e(t)) = \tilde{p}$, it is easy to verify that

$${}^1\mathbf{x}^j(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t)) = \mathbf{w}^j(t) - \sum_{i=1}^l \mathbf{w}^j(t)\lambda^i + \sum_{i=1}^l \mathbf{w}^i(t)\lambda^j \frac{\tilde{p}^i}{\tilde{p}^j} = z^j,\ j = 1, \ldots, l.$$

But then, we have

$$u_t({}^1\mathbf{x}(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t)))) = u_t(z) > u_t(\tilde{\mathbf{x}}(t)) = u_t({}^1\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))),$$

which contradicts the fact that the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}}))$ is a Cournot-Walras equilibrium. (ii) Let $(p^*, \mathbf{x}^*)$ be a Walras equilibrium. First, notice that, by

Assumption 2, $p^* \in R^l_{++}$ and $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$, for all $t \in T$. But then, by Lemma 5 in Codognato and Ghosal (2000a), for all $t \in T_1$, there exist $\lambda^{*j}(t) \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^{l} \lambda^{*j}(t) = 1$, such that

$$\mathbf{x}^{*j}(t) = \lambda^{*j}(t) \frac{\sum_{j=1}^{l} p^{*j}\mathbf{w}^j(t)}{p^{*j}},\ j = 1, \ldots, l.$$

Define now a function $\boldsymbol{\lambda}^* : T_1 \to R^l_+$ such that $\boldsymbol{\lambda}^{*j}(t) = \lambda^{*j}(t)$, $j = 1, \ldots, l$, for all $t \in T_1$ and a function $\mathbf{e}^* : T_1 :\to R^{l^2}_+$ such that $\mathbf{e}^*_{ij}(t) = \mathbf{w}^i(t)\boldsymbol{\lambda}^{*j}(t)$, $i, j = 1, \ldots, l$, for all $t \in T_1$. It is straightforward to show that the function $\mathbf{e}^*$ is integrable. Moreover, by using Equation (2), it is easy to verify that

$$\mathbf{x}^{*j}(t) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{e}^*_{ji}(t) + \sum_{i=1}^{l} \mathbf{e}^*_{ij}(t) \frac{p^{*i}}{p^{*j}},$$

$j = 1, \ldots, l$, for all $t \in T_1$. As $\mathbf{x}^*$ is an allocation, it follows that

$$\int_{T_0} \mathbf{x}^{*j}\, d\mu + \int_{T_1} \mathbf{w}^j(t)\, d\mu - \sum_{i=1}^{l} \int_{T_1} \mathbf{e}^*_{ji}(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}^*_{ij}(t)\, d\mu \frac{p^{*i}}{p^{*j}} = \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1, \ldots, l$. This, in turn, implies that

$$\int_{T_0} \mathbf{x}^{*j}(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}^*_{ij}(t)\, d\mu \frac{p^{*i}}{p^{*j}} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}^*_{ji}(t)\, d\mu,$$

$j = 1, \ldots, l$. But then, by Assumption 4, there is a price selection $p(\mathbf{e})$ such that $p^* = p(\mathbf{e}^*)$ and, consequently, $\mathbf{x}^*(t) = {}^0\mathbf{x}(t, p(\mathbf{e}^*))$, for all $t \in T_0$, and $\mathbf{x}^*(t) = {}^1\mathbf{x}(t, \mathbf{e}^*(t), p(\mathbf{e}^*))$, for all $t \in T_1$. It remains to show that no trader $t \in T_1$ has an advantageous deviation from $\mathbf{e}^*$. Suppose, on the contrary, that there exists a trader $t \in T_1$ and a strategy $e(t) \in \mathbf{E}(t)$ such that

$$u_t({}^1\mathbf{x}(t, , e(t), p(\mathbf{e}^* \setminus e(t)))) > u_t({}^1\mathbf{x}(t, \mathbf{e}^*(t), p(\mathbf{e}^*))).$$

By Equation (1), we have $p(\mathbf{e}^* \setminus e(t)) = p(\mathbf{e}^*) = p^*$. Moreover, it is easy to show that $p^{*1}\mathbf{x}(t, e(t), p(\mathbf{e}^* \setminus e(t)) = p^*\mathbf{w}(t)$. But then, the pair $(p^*, \mathbf{x}^*)$ is not a Walras equilibrium, which generates a contradiction. ■

The following corollary assures the existence of a Cournot-Walras equilibrium in limit exchange economies.

Corollary. *A Cournot-Walras equilibrium exists.*

Proof. From Aumann (1966), we know that, under Assumptions 1, 2, and 3, a Walras equilibrium exists. But then, by part (ii) of Proposition 2, this implies that a Cournot-Walras equilibrium exists. ∎

Proposition 2 shows that noncooperative strategic behaviour is ineffective when strategic traders are represented by an atomless measure space. The question arises whether that proposition still holds in our original framework, where the oligopolists are represented by a purely atomic measure space. Shitovitz (1973) shows that, counterintuitively, this may not be the case if the core of an exchange economy is considered. In particular, he shows that, if the oligopolists are all the same type, that is they have the same endowments and preferences, the core allocations are Walrasian. The following example analyses an exchange economy with two identical atoms facing an atomless continuum of traders and shows that, in this economy, there is a Cournot-Walras equilibrium allocation which is not Walrasian.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4, where* $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, *for all* $t \in T_1$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, *for all* $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = lnx^1 + lnx^2$, *for all* $t \in [\frac{1}{2}, 1]$. *For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any Walras equilibrium.*

Proof. The only symmetric Cournot-Walras equilibrium is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{20+8\sqrt{13}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{10+4\sqrt{13}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{1}{2}$, for all $t \in [\frac{1}{2}, 1]$. On the other hand, the only Walras equilibrium of the economy considered is the pair $(\mathbf{x}^*, p^*)$, where $\mathbf{x}^{*1}(2) = \mathbf{x}^{*1}(3) = \frac{1}{2}$, $\mathbf{x}^{*2}(2) = \mathbf{x}^{*2}(3) = \frac{1}{10}$, $\mathbf{x}^{*1}(t) = \frac{1}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{10}$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{x}^{*1}(t) = \frac{5}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{2}$, for all $t \in [\frac{1}{2}, 1]$, $p^* = \frac{1}{5}$. ∎

The model introduced in Section 2 is a version à la Cournot-Walras of a model first proposed by Lloyd S. Shapley and further analysed by Sahi and Yao (1989) in the case of exchange economies with a finite number of traders. If we consider the mixed version of this model, all traders behave strategically but those belonging to the atomless sector have a negligible influence on prices. The strategic behaviour of the atomless sector could consequently be interpreted as a competitive behaviour. On the other hand, in our version à la Cournot-Walras of the Shapley's model, the atomless sector is supposed to behave competitively while the atoms have strategic power. Therefore, it seems to be reasonable to conjecture that any allocation corresponding to our variant of the mixed version of the Shapley's model correspond to a Cournot-Nash equilibrium of the mixed version of the original Shapley's model. Surprisingly, this conjecture turns out to be false. In order to show it, let us first introduce the mixed version of the original Shapley's model, where the space of traders is as in Section 2 above. Let $b \in R^{l^2}$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll-1}, b_{ll})$. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R^{l^2} : b_{ij} \geq 0,\, i, j = 1, \ldots, l;\, \sum_{j=1}^{l} b_{ij} \leq \mathbf{w}^i(t),\, i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{b} : T \to R^{l^2}$, such that, for all $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. For each $t \in T$, $\mathbf{b}_{ij}(t)$, $i, j = 1, \ldots, l$, is the amount of commodity i that trader t offers in exchange for commodity j. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}$ as $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$. Moreover, we denote by $\mathbf{b} \backslash b(t)$ a strategy selection obtained by replacing $\mathbf{b}(t)$ in $\mathbf{b}$ with $b(t) \in \mathbf{B}(t)$. Then, we are able to introduce the following definition (see Sahi and Yao (1989)).

Definition 2. *Given a strategy selection* $\mathbf{b}$*, a price vector* p *is market clearing if*

$$p \in R^l_{++},\ \sum_{i=1}^{l} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji}),\ j = 1, \ldots, l. \tag{3}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (3) if and only if $\bar{\mathbf{B}}$ is irreducible. Denote by $p(\mathbf{b})$ the function which associates, to each strategy selection $\mathbf{b}$ such that $\bar{\mathbf{B}}$ is irreducible, the unique, up to a scalar multiple, market clearing price vector p. Given a strategy selection $\mathbf{b}$ such that p is market clearing and unique, up to a scalar multiple, consider the assignment determined as follows:

$$\mathbf{x}^j(t, \mathbf{b}(t), p(\mathbf{b})) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i(\mathbf{b})}{p^j(\mathbf{b})},$$

for all $t \in T$, $j = 1, \ldots, l$. It is easy to verify that this assignment is an allocation. Given a strategy selection $\mathbf{b}$, the traders' final holdings are

$$\begin{aligned} \mathbf{x}^j(t) &= \mathbf{x}^j(t, \mathbf{b}(t), p(\mathbf{b})) \text{ if } p \text{ is market clearing and unique,} \\ \mathbf{x}^j(t) &= \mathbf{w}^j(t) \text{ otherwise,} \end{aligned}$$

for all $t \in T$, $j = 1, \ldots, l$. This respecification of the Shapley's model allows us to define the following concept of Cournot-Nash equilibrium for exchange economies with a continuum of traders (see Codognato and Ghosal (2000a)).

Definition 3. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for all $t \in T$ and for all $b(t) \in \mathbf{B}(t)$.

Codognato and Ghosal (2000b) show that, under some further assumptions on the atoms, a Cournot-Nash equilibrium exists. Moreover, Codognato and Ghosal (2000a) show that, in limit exchange economies, the set of the Cournot-Nash equilibrium allocations coincides with the set of the Walras equilibrium allocations. But then, taking into account Proposition 2, we can conclude that, in limit exchange economies, the set of the Cournot-Walras, Cournot-Nash, and Walras equilibrium allocations are equivalent. We provide now an example which shows that, in mixed exchange economies, there is a Cournot-Walras equilibrium allocation which does not correspond to any Cournot-Nash equilibrium.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4, where $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in T_1$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = lnx^1 + lnx^2$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = x^1 + lnx^2$, for all $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any Cournot-Nash equilibrium.*

Proof. The only symmetric Cournot-Walras equilibrium of the economy considered is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{-1+\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11-\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{-1+\sqrt{37}}{14+4\sqrt{37}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{7+2\sqrt{37}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{1+2\sqrt{37}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{6}{7+2\sqrt{37}}$, for all $t \in [\frac{1}{2}, 1]$. On the other hand, the only symmetric Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{b}}_{12}(2) = \hat{\mathbf{b}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\hat{\mathbf{b}}_{12}(t) = \frac{1}{2}$, for all $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{b}}_{21}(t) = \frac{5+2\sqrt{13}}{11+2\sqrt{13}}$ for all $t \in [\frac{1}{2}, 1]$, which generates the allocation $\hat{\mathbf{x}}^1(2) = \hat{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\hat{\mathbf{x}}^2(2) = \hat{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{22+4\sqrt{13}}$, $\hat{\mathbf{x}}^1(t) = \frac{1}{2}$, $\hat{\mathbf{x}}^2(t) = \frac{3}{11+2\sqrt{13}}$, for all $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\hat{\mathbf{x}}^2(t) = \frac{6}{11+2\sqrt{13}}$, for all $t \in [\frac{1}{2}, 1]$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}, p(\hat{\mathbf{b}}))$, for all $t \in T$. ■

We proceed now to investigate the relationship between the Cournot-Walras equilibrium concept introduced in this paper and the Cournot-Walras equilibrium concept for exchange economies previously proposed by Codognato and Gabszewicz (1991). The latter concept has been generalized by Gabszewicz and Michel (1997) by using a notion of oligopoly equilibrium for exchange economies (see also d'Aspremont et al. (1997), for another generalization of that concept). More precisely, the Cournot-Walras equilibrium introduced by Codognato and Gabszewicz (1991) corresponds to the case of homogeneous oligopoly equilibrium in the framework developed by Gabszewicz and Michel (1997) and can be formulated as follows. Consider a particular specification of the mixed exchange economy defined in Section 2, where the initial assignment of atoms is $\mathbf{w}(t) = (\mathbf{w}^1(t), 0, \ldots, 0)$, for all $t \in T_1$. A strategy correspondence is a correspondence $\mathbf{Y} : T_1 \to \mathcal{P}(R)$ such that, for each $t \in T_1$, $\mathbf{Y}(t) = \{y \in R : 0 \leq y \leq \mathbf{w}^1(t)\}$. A strategy selection is an integrable function $\mathbf{y} : T_1 \to R$ such that, for all $t \in T_1$, $\mathbf{y}(t) \in \mathbf{Y}(t)$. For each $t \in T_1$, $\mathbf{y}(t)$ is the amount of commodity 1 that trader t offers in the market. We denote by $\mathbf{y} \setminus y(t)$ a strategy selection obtained by replacing $\mathbf{y}(t)$ in $\mathbf{y}$ with $y(t) \in \mathbf{Y}(t)$. As for traders $t \in T_0$, let the function ${}^0\mathbf{x}(\cdot, p)$ be defined as in Section 2. As for traders $t \in T_1$, given a price vector $p \in R^l_{++}$ and a strategy selection $\mathbf{y}$, let ${}^1\mathbf{x}(\cdot, \mathbf{y}(\cdot), p) : T_1 \to R^l_+$ denote a function such that, for each $t \in T_1$, ${}^1\mathbf{x}^1(t, \mathbf{y}(t), p) = \mathbf{w}^1(t) - \mathbf{y}(t)$ and $({}^1\mathbf{x}^2(t, \mathbf{y}(t), p), \ldots, {}^1\mathbf{x}^l(t, \mathbf{y}(t), p))$ is, under Assumption 2, the unique solution to the problem

$$\max_{x^2, \ldots, x^l} u_t(\mathbf{w}^1(t) - \mathbf{y}(t), x^2, \ldots, x^l) s.t. \sum_{j=2}^{l} p^j x^j = p^1 \mathbf{y}(t).$$

Let $\pi(\mathbf{y})$ denote the correspondence which associates, to each strategy selection $\mathbf{y}$, the set of the price vectors such that

$$\int_{T_0} {}^0\mathbf{x}^1(t, p)\, d\mu = \int_{T_0} {}^0\mathbf{w}^1(t)\, d\mu + \int_{T_1} \mathbf{y}(t)\, d\mu,$$

$$\int_{T_0} {}^0\mathbf{x}^j(t, p)\, d\mu + \int_{T_1} {}^1\mathbf{x}^j(t, \mathbf{y}(t), p)\, d\mu = \int_{T_0} {}^0\mathbf{w}^j(t)\, d\mu,$$

$j = 2, \ldots, l$. We assume that, for each $\mathbf{y}$, $\pi(\mathbf{y}) \neq \emptyset$ and $\pi(\mathbf{y}) \subset R^l_{++}$. A price selection $p(\mathbf{y})$ is a function which associates, to each $\mathbf{y}$, a price vector $p \in \pi(\mathbf{y})$. Given a strategy selection $\mathbf{y}$, by the structure of the traders' measure space, Proposition 1, and the atoms' maximization problem, it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = {}^0\mathbf{x}(t, p(\mathbf{y}))$, for all $t \in T_0$, and $\mathbf{x}(t) = {}^1\mathbf{x}(t, \mathbf{y}(t), p(\mathbf{y}))$, for all $t \in T_1$, is an allocation. At this stage, we are able to define the concept of homogeneous oligopoly equilibrium.

Definition 4. *A pair $(\check{\mathbf{y}}, \check{\mathbf{x}})$, consisting of a strategy selection $\check{\mathbf{y}}$ and an allocation $\check{\mathbf{x}}$ such that $\check{\mathbf{x}}(t) = {}^0\mathbf{x}(t, p(\check{\mathbf{y}}))$, for all $t \in T_0$, and $\check{\mathbf{x}}(t) = {}^1\mathbf{x}(t, \mathbf{y}(t), p(\mathbf{t}))$, for all $t \in T_1$, is a homogeneous oligopoly equilibrium, with respect to a price selection $p(\mathbf{y})$, if $u_t({}^1\mathbf{x}(t, \check{\mathbf{y}}(t), p(\check{\mathbf{y}}))) \geq u_t({}^1\mathbf{x}, y(t), (t, p(\check{\mathbf{y}} \setminus y(t))))$, for all $t \in T_1$ and for all $y(t) \in \mathbf{Y}(t)$.*

By means of the following example, we prove now that there is a Cournot-Walras equilibrium allocation which does not correspond to any oligopoly equilibrium.

Example 3. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4, where $l = 3$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0, 0)$, $u_t(x) = 2x^1 + lnx^2 + lnx^3$, for all $t \in T_1$, $\mathbf{w}(t) = (1, 0, 0)$, $u_t(x) = lnx^1 + lnx^2 + lnx^3$, for all $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1, 1)$, $u_t(x) = x^1 + \frac{1}{2}lnx^2 + lnx^3$, for all $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any oligopoly equilibrium.*

Proof. There is a unique Cournot-Walras equilibrium represented by the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{1+\sqrt{241}}{48}$, $\tilde{\mathbf{e}}_{13}(2) = \tilde{\mathbf{e}}_{13}(3) = \frac{-1+\sqrt{97}}{24}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{49-\sqrt{241}-2\sqrt{97}}{48}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{1+\sqrt{241}}{44+4\sqrt{241}}$, $\tilde{\mathbf{x}}^3(2) = \tilde{\mathbf{x}}^3(3) = \frac{-1+\sqrt{97}}{28+4\sqrt{97}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{3}$, $\tilde{\mathbf{x}}^2(t) = \frac{4}{11+\sqrt{241}}$, $\tilde{\mathbf{x}}^3(t) = \frac{2}{7+\sqrt{97}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{9+\sqrt{97}+\sqrt{241}}{12}$, $\tilde{\mathbf{x}}^2(t) = \frac{6}{11+\sqrt{241}}$, $\tilde{\mathbf{x}}^3(t) = \frac{6}{7+\sqrt{97}}$, for all $t \in [\frac{1}{2}, 1]$. On the other hand, there is no interior symmetric homogeneous oligopoly equilibrium for the economy considered. ■

4 Cournot-Walras equilibrium as a subgame perfect equilibrium

Example 2 above shows that, in mixed exchange economies, there is a Cournot-Walras equilibrium allocation which does not correspond to any Cournot-

Nash equilibrium. This raises the question whether there may exist some other game theoretical framework where the Cournot-Walras equilibrium allocations can be supported by an appropriate equilibrium concept. As the nonequivalence holds in a one-stage game, we are led to consider a multistage game. In particular, as the Cournot-Walras equilibrium concept introduced in Section 2 has an intrinsic two-stage flavour, it seems to be natural to consider a two-stage game where the atoms move in the first stage and the atomless sector moves in the second stage, after observing the first stage atoms' moves. The main theorem of this paper shows that any Cournot-Walras equilibrium allocation corresponds to a subgame perfect equilibrium of the game just sketched. More precisely, we consider the same mixed exchange economy as in Section 2. To this economy, we associate a two-stage game with observed actions (see Fudenberg and Tirole (1991)), which represents a sequential reformulation of the mixed version of the Shapley's model, analysed in Section 3. Let $a \in R^{l^2}$ be a vector such that $a = (a_{11}, a_{12}, \ldots, a_{ll-1}, a_{ll})$. The game is played in two stages, 0 and 1. We denote by $\mathbf{A}^0$ an action correspondence in stage 0, defined on T, such that $\mathbf{A}^0(t)$ is the singleton "do nothing," for all $t \in T_0$, and $\mathbf{A}^0(t) = \{a \in R^{l^2} : a_{ij} \geq 0,\, i, j = 1, \ldots, l;\, \sum_{j=1}^{l} a_{ij} \leq \mathbf{w}^i(t),\, i = 1, \ldots, l\}$, for all $t \in T_1$. We denote by $\mathbf{A}^1$ an action correspondence in stage 1, defined on T, such that $\mathbf{A}^1(t) = \{a \in R^{l^2} : a_{ij} \geq 0,\, i, j = 1, \ldots, l;\, \sum_{j=1}^{l} a_{ij} \leq \mathbf{w}^i(t),\, i = 1, \ldots, l\}$, for all $t \in T_0$, and $\mathbf{A}^1(t)$ is the singleton "do nothing," for all $t \in T_1$. An action selection in stage 0 is a function $\mathbf{a}^0$, defined on T, such that $\mathbf{a}^0(t) \in \mathbf{A}^0(t)$, for all $t \in T$, and ${}^1\mathbf{a}^0$ is integrable. For each $t \in T_1$, ${}^1\mathbf{a}^0(t)$, $i, j = 1, \ldots, l$, is the amount of commodity i that trader t offers in exchange for commodity j. An action selection in stage 1 is a function $\mathbf{a}^1$, defined on T, such that $\mathbf{a}^1(t) \in \mathbf{A}^1(t)$, for all $t \in T$, and ${}^0\mathbf{a}^1$ is integrable. For each $t \in T_0$, ${}^0\mathbf{a}^1(t)$, $i, j = 1, \ldots, l$, is the amount of commodity i that trader t offers in exchange for commodity j. Let S^0 and S^1 be the sets of all action selections in stage 0 and stage 1, respectively, and let H^0 and H^1 be the sets of all stage 0 and stage 1 histories, respectively, where $H^0 = \emptyset$ and $H^1 = S^0$. In addition, let $H^2 = S^0 \times S^1$ be the set of all final histories. Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, we define the aggregate matrix $\bar{\mathbf{A}}$ as $\bar{\mathbf{A}} = (\bar{\mathbf{a}}_{ij}) = (\int_{T_0} {}^0\mathbf{a}^1_{ij}(t)\, d\mu + \int_{T_1} {}^1\mathbf{a}^0{}_{ij}(t)\, d\mu)$. Then, we can introduce the following definition (see Sahi and Yao (1989)).

Definition 5. *Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, a price vector p is market* *clearing if*

$$p \in R^l_{++}, \sum_{i=1}^{l} p^i \bar{\mathbf{a}}_{ij} = p^j (\sum_{i=1}^{l} \bar{\mathbf{a}}_{ji}), \; j = 1, \ldots, l. \tag{4}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (4) if and only if $\bar{\mathbf{A}}$ is irreducible. Denote by $p(\mathbf{h}^2)$ the function which associates, to each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that $\bar{\mathbf{A}}$ is irreducible, the unique, up to a scalar multiple, market clearing price vector p. Given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that p is market clearing and unique, up to a scalar multiple, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} {}^0\mathbf{a}^1_{ji}(t) + \sum_{i=1}^{l} {}^0\mathbf{a}^1_{ij}(t) \frac{p^i(\mathbf{h}^2)}{p^j(\mathbf{h}^2)}, \text{ for all } t \in T_0, \\ \mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} {}^1\mathbf{a}^0_{ji}(t) + \sum_{i=1}^{l} {}^1\mathbf{a}^0_{ij}(t) \frac{p^i(\mathbf{h}^2)}{p^j(\mathbf{h}^2)}, \text{ for all } t \in T_1, \end{aligned} \tag{5}$$

$j = 1, \ldots, l$. It is easy to verify that this assignment is an allocation. Finally, given a final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, the traders' final holdings are

$$\begin{aligned} \mathbf{x}^j(t) &= \mathbf{x}^j(t, \mathbf{h}^2(t), p(\mathbf{h}^2)) \text{ if } p \text{ is market clearing and unique}, \\ \mathbf{x}^j(t) &= \mathbf{w}^j(t) \text{ otherwise}, \end{aligned} \tag{6}$$

for all $t \in T$, $j = 1, \ldots, l$. Now, we define a strategy profile, $\mathbf{s}$, as a sequence of functions $\{\mathbf{s}^0, \mathbf{s}^1\}$, where $\mathbf{s}^0$ is defined on $T \times H^0$ and such that, given $\mathbf{h}^0 \in H^0$, $\mathbf{s}^0(t, \mathbf{h}^0) \in \mathbf{A}^0(t)$, for all $t \in T$, and $\mathbf{s}^0(\cdot, \mathbf{h}^0) \in S^0$; $\mathbf{s}^1$ is defined on $T \times H^1$ and such that, given $\mathbf{h}^1 \in H^1$, $\mathbf{s}^1(t, \mathbf{h}^1) \in \mathbf{A}^1(t)$, for all $t \in T$, $\mathbf{s}^1(\cdot, \mathbf{h}^1) \in S^1$. Moreover, we denote by $\mathbf{s} \setminus s(t, \cdot) = \{\mathbf{s}^0 \setminus s^0(t, \cdot), \mathbf{s}^1 \setminus s^1(t, \cdot)\}$ a strategy profile obtained by replacing $\mathbf{s}^0(t, \cdot)$ in $\mathbf{s}^0$ and $\mathbf{s}^1(t, \cdot)$ in $\mathbf{s}^1$, respectively, with the functions $s^0(t, \cdot)$ and $s^1(t, \cdot)$, where $s^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$ and $s^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$. With a little abuse of notation, given a strategy profile $\mathbf{s}$, we denote by ${}^1\mathbf{s}^0$ and ${}^0\mathbf{s}^1$ the functions defined, respectively, on T_1 and T_0, such that ${}^1\mathbf{s}^0(t) = {}^1\mathbf{a}^0(t) = \mathbf{s}^0(t, \mathbf{h}^0)$, for all $t \in T_1$, and ${}^0\mathbf{s}^1(t) = {}^0\mathbf{a}^1(t) = \mathbf{s}^1(t, \mathbf{h}^1)$, for all $t \in T_0$, with $\mathbf{h}^1 = \mathbf{s}^0(\cdot, \mathbf{h}^0)$. In addition, given a strategy profile $\mathbf{s}$, we define the aggregate matrix $\bar{\mathbf{S}}$ as $\bar{\mathbf{S}} = (\bar{\mathbf{s}}_{ij}) = (\int_{T_0} {}^0\mathbf{s}^1{}_{ij}(t)\, d\mu + \int_{T_1} {}^1\mathbf{s}^0{}_{ij}(t)\, d\mu)$. Then, given a strategy profile $\mathbf{s}$ such that $\bar{\mathbf{S}}$ is irreducible, we denote by $p(\mathbf{s})$ the function obtained by replacing, in Equation (4), $\bar{\mathbf{a}}_{ij}$ with $\bar{\mathbf{s}}_{ij}$, $i, j = 1, \ldots, l$. Given a strategy profile $\mathbf{s}$ such that p is market clearing and unique, up to a scalar multiple, the allocation $\mathbf{x}(t, \mathbf{s}(t), p(\mathbf{s}))$ is obtained by replacing, in (5), $\mathbf{h}^2$ with $\mathbf{s}$ and ${}^0\mathbf{a}^1$, ${}^1\mathbf{a}^0$, respectively, with ${}^0\mathbf{s}^1$, ${}^1\mathbf{s}^0$. Finally, the traders' final holdings are determined as in (6), by replacing $\mathbf{h}^2$ with $\mathbf{s}$. We proceed now to consider the subgame consisting of the stage 1 of the game outlined above, given the history $\mathbf{h}^1 \in H^1$. Given a strategy profile $\mathbf{s}$, the strategy selection $\mathbf{s}|\mathbf{h}^1$ is a function, defined on T, such that, for each $\mathbf{h}^1 \in H^1$, $\mathbf{s}(t)|\mathbf{h}^1 = \mathbf{s}^1(t, \mathbf{h}^1)$, for all $t \in T$. In addition, given a history $\mathbf{h}^1 \in H^1$ and a strategy profile $\mathbf{s}$, we define the aggregate matrix $\bar{\mathbf{S}}|\mathbf{h}^1$ as $\bar{\mathbf{S}}|\mathbf{h}^1 = (\bar{\mathbf{s}}_{ij}|\mathbf{h}^1) = (\int_{T_0} {}^0\mathbf{s}^1{}_{ij}(t)|\mathbf{h}^1 \, d\mu + \int_{T_1} {}^1\mathbf{h}^1{}_{ij}(t) \, d\mu)$. Then, given a history $\mathbf{h}^1 \in H^1$, and a strategy profile $\mathbf{s}$ such that $\bar{\mathbf{S}}|\mathbf{h}^1$ is irreducible, we denote by $p(\mathbf{s}|\mathbf{h}^1)$ the function obtained by replacing, in Equation (4), $\bar{\mathbf{a}}_{ij}$ with $\bar{\mathbf{s}}_{ij}|\mathbf{h}^1$, $i, j = 1, \ldots, l$. Given a history $\mathbf{h}^1 \in H^1$ and a strategy profile $\mathbf{s}$ such that p is market clearing and unique, up to a scalar multiple, the allocation $\mathbf{x}(t, \mathbf{s}|\mathbf{h}^1(t), p(\mathbf{s}|\mathbf{h}^1))$ is obtained by replacing, in (5), $\mathbf{h}^2$ by $\mathbf{s}|\mathbf{h}^1$ and ${}^0\mathbf{a}^1$, ${}^1\mathbf{a}^0$, respectively, with ${}^0\mathbf{s}^1|\mathbf{h}^1$, ${}^1\mathbf{h}^1$. The traders' final holdings are determined as in (6), by replacing $\mathbf{h}^2$ with $\mathbf{s}|\mathbf{h}^1$. Finally, given a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1$ a strategy selection obtained by replacing $\mathbf{s}(t)|\mathbf{h}^1$ in $\mathbf{s}|\mathbf{h}^1$ with $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$. We are now able to define the concept of subgame perfect equilibrium for the two stage game above.

Definition 6. *A strategy profile $\hat{\mathbf{s}}$ such that $\bar{\hat{\mathbf{S}}}|\mathbf{h}^1$ is irreducible, for each $\mathbf{h}^1 \in H^1$, is a subgame perfect equilibrium if, for all $t \in T$,*

$$u_t(\mathbf{x}(t, \hat{\mathbf{s}}(t), p(\hat{\mathbf{s}}))) \geq u_t(\mathbf{x}(t, s(t, \cdot), p(\hat{\mathbf{s}} \setminus s(t, \cdot)))),$$

for all possible functions $s(t, \cdot)$, and, for each $\mathbf{h}^1 \in H^1$,

$$u_t(\mathbf{x}(t, \hat{\mathbf{s}}|\mathbf{h}^1(t), p(\hat{\mathbf{s}}|\mathbf{h}^1))) \geq u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\hat{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1))),$$

for all $t \in T$ and for all $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$.

In order to prove our main theorem, we need to introduce a further assumption on the atomless sector. We denote by L the set of commodities $\{1, \ldots, l\}$ and by $R^l_{+j>0} \subset R^l_+$ the set of vectors in R^l_+, whose j-th component is strictly positive. For each $i \in L$, we consider the set $T_i = \{t \in T_0 : \mathbf{w}^i(t) > 0\}$. Clearly, by Assumption 1, $\mu(T_i) > 0$. We say that the commodities $i, j \in L$

stand in the relation C if there is a measurable subset T_i' of T_i, with $\mu(T_i') > 0$, such that, for each trader $t \in T_i'$, $\{x \in R^l_+ : u_t(x) = u_t(y)\} \subset R^l_{+j>0}$, for all $y \in R^l_{++}$. In addition, we use the following definition provided by Codognato and Ghosal (2000a), to whom we refer for further details.

Definition 7. *The set of commodities L is said to be a net if $\{\langle i, j\rangle : iCj\} \neq \emptyset$ and the directed graph $D_L(L, C)$ is strongly connected.*

Then, we can introduce the following assumption.

Assumption 5. *The set of commodities L is a net.*

We are now ready to state and prove the main theorem.

Theorem. *Under Assumptions 1, 2, 3, 4 and 5, if $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot-Walras equilibrium with respect to a price selection $p(\mathbf{e})$, there is a subgame perfect equilibrium $\tilde{\mathbf{s}}$ such that $\mathbf{x}(t, p(\tilde{\mathbf{e}})) = \mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T$.*

Proof. Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot-Walras equilibrium with respect to the price selection $p(\mathbf{e})$. Let $p(\mathbf{h}^1)$ denote a function obtained by replacing, in the price selection $p(\mathbf{e})$, each strategy selection $\mathbf{e}$ with a history $\mathbf{h}^1$ such that $\mathbf{h}^1(t) = \mathbf{e}(t)$, for all $t \in T_1$. Consider now a history $\mathbf{h}^1 \in H^1$. As, by Assumption 4, $p(\mathbf{h}^1) \gg 0$, Assumption 2 implies that $p(\mathbf{h}^1) {}^0\mathbf{x}(t, p(\mathbf{h}^1)) = p(\mathbf{h}^1)\mathbf{w}(t)$, for all $t \in T_0$. But then, by Lemma 5 in Codognato and Ghosal (2000a), for all $t \in T_0$, there exist $\lambda^j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^j = 1$, such that

$$^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \lambda^j \frac{\sum_{j=1}^l p^j(\mathbf{h}^1)\mathbf{w}^j(t)}{p^j(\mathbf{h}^1)},\ j = 1, \ldots, l.$$

Define now a function $\boldsymbol{\lambda} : T_0 \to R^l_+$ such that $\boldsymbol{\lambda}^j(t) = \lambda^j(t)$, $j = 1, \ldots, l$, for all $t \in T_0$. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a function, defined on T, such that $\tilde{\mathbf{s}}(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$, for all $t \in T$, and such that ${}^0\tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1 = \mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)$, $i, j = 1, \ldots, l$, for all $t \in T_0$. It is straightforward to show that the function ${}^0\tilde{\mathbf{s}}|\mathbf{h}^1$ is integrable. We want now to show that the matrix $\bar{\tilde{\mathbf{S}}}|\mathbf{h}^1 = (\bar{\tilde{\mathbf{s}}}_{ij}|\mathbf{h}^1) = (\int_{T_0} {}^0\mathbf{s}^1{}_{ij}(t)|\mathbf{h}^1\, d\mu + \int_{T_1} {}^1\mathbf{h}^1_{ij}(t)\, d\mu)$ is irreducible. Let $i, j \in L$ be two commodities which stand in the relation C. Consider a trader $t \in T_i'$. First, observe that $p(\mathbf{h}^1)\mathbf{w}(t) > 0$ since, by Assumption 4, $p(\mathbf{h}^1) \gg 0$. This, together with Assumption 2, implies that ${}^0\mathbf{x}(t, p(\mathbf{h}^1)) > 0$ and, since the commodities i and j stand in the realtion C, that ${}^0\mathbf{x}^j(t, p(\mathbf{h}^1)) > 0$. Consider now the matrix $\bar{\mathbf{S}}^L|\mathbf{h}^1 = (\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1)$ such that $\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 = \int_{T_i'} \mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)\, d\mu$, if iCj, $\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 = 0$, otherwise. If iCj,

$\bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1 > 0$, since, for each $t \in T'_i$, $\mathbf{w}^i(t) > 0$ and, by the above argument, $\boldsymbol{\lambda}^j(t) > 0$. But then, the matrix $\bar{\bar{\mathbf{S}}}|\mathbf{h}^1$ is irreducible as it is such that $\bar{\bar{\mathbf{s}}}_{ij}|\mathbf{h}^1 \geq \bar{\mathbf{s}}^L_{ij}|\mathbf{h}^1$, $i, j = 1, \ldots, l$, and the matrix $\bar{\mathbf{S}}^L|\mathbf{h}^1$, by Assumption 5 and by the argument used in the proof of Theorem 2 in Codognato and Ghosal (2000a), is irreducible. As it is easy to verify that

$$^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \tilde{\mathbf{s}}_{ji}(t)|\mathbf{h}^1 + \sum_{i=1}^{l} \tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1 \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)},$$

for all $t \in T_0$, $j = 1, \ldots, l$, and as $p(\mathbf{h}^1)$ satisfies Equation (1), we have

$$\int_{T_0} \mathbf{w}^j(t)\, d\mu - \sum_{i=1}^{l} \int_{T_0} \tilde{\mathbf{s}}_{ji}(t)|\mathbf{h}^1\, d\mu + \sum_{i=1}^{l} \int_{T_0} \tilde{\mathbf{s}}_{ij}(t)|\mathbf{h}^1\, d\mu \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)}$$
$$+ \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ij}(t)\, d\mu \frac{p^i(\mathbf{h}^1)}{p^j(\mathbf{h}^1)} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{h}^1_{ji}(t)\, d\mu,$$

$j = 1, \ldots, l$. This implies that

$$\sum_{i=1}^{l} p^i(\mathbf{h}^1)\bar{\bar{\mathbf{s}}}_{ij}|\mathbf{h}^1 = p^j(\mathbf{h}^1)(\sum_{i=1}^{l} \bar{\bar{\mathbf{s}}}_{ji}|\mathbf{h}^1),\ j = 1, \ldots, l,$$

and, thus, by Equation (4), that $p(\mathbf{h}^1) = p(\tilde{\mathbf{s}}|\mathbf{h}^1)$. It is then straightforward to verify that $^0\mathbf{x}^j(t, p(\mathbf{h}^1)) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1))$, for all $t \in T_0$, $j = 1, \ldots, l$, $^1\mathbf{x}^j(t, \mathbf{h}^1(t), p(\mathbf{h}^1)) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1))$, for all $t \in T_1$, $j = 1, \ldots, l$. It remains now to show that no trader $t \in T$, in stage 1, has an advantageous deviation from $\tilde{\mathbf{s}}(t)|\mathbf{h}^1$. This is trivially true for all $t \in T_1$. Suppose now that there exist a trader $t \in T_0$ and an action $s(t)|\mathbf{h}^1 \in \mathbf{A}^1(t)$ such that

$$u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1))) > u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t)|\mathbf{h}^1, p(\mathbf{s}|\mathbf{h}^1))).$$

Since, as an immediate consequence of Definition 5, $p(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s(t)|\mathbf{h}^1) = p(\tilde{\mathbf{s}}|\mathbf{h}^1)$, the above argument and the former inequality imply that

$$u_t(\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\mathbf{h}^1))) > u_t(^0\mathbf{x}(t, p(\mathbf{h}^1))).$$

As it is easy to check that $p(\mathbf{h}^1)\mathbf{x}(t, s(t)|\mathbf{h}^1, p(\mathbf{h}^1)) = p(\mathbf{h}^1)\mathbf{w}(t)$, this, in turn, implies that $^0\mathbf{x}(t, p(\mathbf{h}^1)) \notin \boldsymbol{\Delta}_{p(\mathbf{h}^1)}(t) \cap \boldsymbol{\Gamma}_{p(\mathbf{h}^1)}(t)$, a contradiction. Let now $\tilde{\mathbf{h}}^1$ be a history such that $\tilde{\mathbf{h}}^1(t) = \tilde{\mathbf{e}}(t)$, for all $t \in T_1$, and let $\tilde{\mathbf{s}}$ be a strategy profile such that, for all $t \in T$, $\tilde{\mathbf{s}}^0(t, \mathbf{h}^0) = \tilde{\mathbf{h}}^1(t)$ and $\tilde{\mathbf{s}}^1(t, \mathbf{h}^1) = \tilde{\mathbf{s}}(t)|\mathbf{h}^1$, for each $\mathbf{h}^1 \in H^1$. Then, the above argument implies that $p(\tilde{\mathbf{e}}) = p(\tilde{\mathbf{s}})$ and ${}^0\mathbf{x}^j(t, p(\tilde{\mathbf{e}})) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T_0$, $j = 1, \ldots, l$, ${}^1\mathbf{x}^j(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}})) = \mathbf{x}^j(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))$, for all $t \in T$, $j = 1, \ldots, l$. It remains now to show that no trader $t \in T$ has an advantageous deviation from $\tilde{\mathbf{s}}$. As, for each trader $t \in T_0$, $p(\tilde{\mathbf{s}} \setminus s(t, \cdot)) = p(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus s(t, \tilde{\mathbf{h}}^1)|\tilde{\mathbf{h}}^1)$, it is a straightforward consequence of the previous discussion that no trader $t \in T_0$ has an advantageous deviation from $\tilde{\mathbf{s}}$. Suppose now that there exists a trader $t \in T_1$ and functions $s^0(t, \cdot)$ and $s^1(t, \cdot)$ such that

$$u_t(\mathbf{x}(t, \tilde{\mathbf{s}} \setminus s(t, \cdot), p(\tilde{\mathbf{s}} \setminus s(t, \cdot)))) > u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))).$$

Let $\tilde{\mathbf{h}}^1 \setminus h(t)$ be a history obtained by replacing $\tilde{\mathbf{h}}^1(t)$ in $\tilde{\mathbf{h}}^1$ with $h(t) = s^0(t, \mathbf{h}^0)$ and let $\tilde{\mathbf{e}} \setminus e(t)$ be a strategy selection obtained by replacing $\tilde{\mathbf{e}}(t)$ in $\tilde{\mathbf{e}}$ by $e(t) = s^0(t, \mathbf{h}^0)$. As $p(\tilde{\mathbf{e}} \setminus e(t)) = p(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus h(t)) = p(\tilde{\mathbf{s}} \setminus s(t, \cdot))$, the former inequality implies that

$$\begin{aligned} &u_t({}^1\mathbf{x}(t, e(t), p(\tilde{\mathbf{e}} \setminus e(t)))) = u_t(\mathbf{x}(t, s(t, \cdot), p(\tilde{\mathbf{s}} \setminus s(t, \cdot)))) > \\ &u_t(\mathbf{x}(t, \tilde{\mathbf{s}}(t), p(\tilde{\mathbf{s}}))) = u_t({}^1\mathbf{x}(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))), \end{aligned}$$

which is a contradiction. ■

5 Conclusion

In this paper, we have introduced a respecification à la Cournot-Walras of the mixed version of a model of noncooperative exchange, originally proposed by Lloyd S. Shapley. We have first shown that there is an allocation corresponding to the Cournot-Walras equilibrium of our variant of the mixed version of the Shapley's model which does not correspond to any Cournot-Nash equilibrium of the mixed version of the original Shapley's model. For this reason, we considered a further reformulation of this model as a two-stage game where the atoms move in the first stage and the atomless sector moves in the second stage. Our main result shows that any Cournot-Walras equilibrium allocation corresponds to a subgame perfect equilibrium of this two-stage game. The converse of this result does not hold since, at a subgame perfect equilibrium, the subgames associated with the atoms' strategies leading to the same aggregate bids may be played in different ways. A possible way to avoid this unreasonable behaviour is to consider the so called “Markov strategies,” which depend only on payoff-relevant past events, and to define a notion of Markov perfect equilibium as a subgame perfect equilibrium in which all players use Markov strategies (see Maskin and Tirole (2001)). A further step in future research should go in the direction of showing an equivalence theorem between the set of the Cournot-Walras equilibrium allocations and the set of the Markov perfect equilibrium allocations.

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