On the superiority of the Borda rule in a distance-based perspective on Condorcet efficiency

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Abstract

The Condorcet efficiency of three widely used scoring rules is investigated with the help of computer simulations for larger numbers of candidates than are usually considered in the literature and it is shown that for this case the superiority of the Borda rule does not hold. A new distance-based measure of Condorcet efficiency is introduced which extends the superiority of the Borda rule to larger numbers of candidates.

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1 Introduction

One of the intuitively most appealing principles for collective choice is the criterion associated with the name of Condorcet. The Condorcet criterion requires that a candidate that defeats every other candidate in pairwise majority voting should be elected. The efficiency of voting rules with respect to this type of criterion has been the subject of a whole strand of literature starting with Fishburn (1974) and Fishburn and Gehrlein (1976). In this framework Condorcet efficiency is measured as the conditional probability that a certain voting rule selects the Condorcet winner given that such a candidate exists. Analytical results (e.g. Gehrlein and Fishburn (1978), Newenhizen (1992), Tataru and Merlin (1997)) as well as computer simulations (e.g. Merrill (1984), Vandercruyssen (1999), Lepelley et al. (2000)) have in particular established the superiority of the Borda rule over other scoring rules with respect to Condorcet efficiency.

However, computer simulations in this paper show that this superiority does not hold for a weak version of the Condorcet criterion if one allows for larger numbers of candidates than usual in the literature. Hence we address this issue with a distance-based measure of Condorcet efficiency (instead of a probabilistic measure) for three important scoring rules. It is shown that this distance-based measure extends the superiority of the Borda rule also to larger numbers of candidates.

Scoring rules such as the Borda rule constitute a particular class of social choice rules which operate by computing for each candidate a score that depends on the rank of this candidate in the individual's preference order. A major significance of scoring rules lies in the fact that since they assign a number to each candidate they guarantee a weak order (i.e. a complete, reflexive and transitive social preference) for every profile of individual preferences. Thus scoring rules circumvent a major problem of other social choice procedures, namely the possibility of voting cycles.

The three most widely discussed scoring rules are the plurality rule, the antiplurality rule and the Borda rule. The plurality rule assigns one point to the individual's top ranked candidate and zero to every other candidate, while the anti-plurality rule assigns zero to the individual's bottom ranked candidate and one point to every other candidate. In contrast to these two polar cases, the Borda rule assigns to each of m candidates scores with equal intervals of length $1/(m-1)$ starting with a score of 1 for the individual's top ranked candidate down to a score of 0 for the individual's bottom ranked candidate.

Until now the literature on Condorcet efficiency has not systematically taken into account a source of information that has recently attracted attention in related contexts, namely the distance information contained in the differences in the rankings derived from different voting rules (Saari (2000), Ratliff (2001, 2002), Klamler (2003)).

Distance information is, however, of obvious relevance for the evaluation of different voting rules. Actually, the superiority of the Borda rule is often implicitly justified in terms of distances. In particular it is a widely adduced justification of the Borda rule that the Condorcet winner is never ranked last in the Borda ranking whereas this can be the case in the rankings derived from any other scoring rule (Fishburn and Gehrlein (1986)).

In this paper we measure the efficiency of a given scoring rule with respect to the Condorcet criterion with the help of distances. Previous efficiency concepts with respect to Condorcet criteria are defined by the probability that a certain voting rule selects the Condorcet winner given that such a candidate exists. Thus for every profile the criterion is either satisfied or violated without considering the degree of violation. In this paper, to capture the degree of violation of the Condorcet criterion, we use a distance-based measure for the efficiency of typical scoring rules. In cases where a particular scoring rule fails to elect the Condorcet winner, a natural extension of efficiency considerations leads to the question whether it elects candidates that are at least closer to being the Condorcet winner than the candidates elected by other scoring rules. This approach will turn out particularly significant, if, in contrast to most of the literature on the Condorcet efficiency of voting rules, the number of candidates is allowed to grow large. This could, for example, be the case for recruitment committees having to rank a large number of candidates for a limited number of job openings.

The paper is structured as follows. Section 2 introduces the formal framework, section 3 presents and discusses the results and section 4 briefly concludes the paper. Tables and graphs of the computer simulations are provided in the Appendix.[1]

2 Formal framework

Let X denote a finite set of m candidates and H denote a finite set of n voters. A preference $R \subseteq X \times X$ is a binary relation on X. The symmetric and asymmetric parts of R will be denoted by $\sim_R$ and $\succ_R$ respectively. The preference of voter $k \in H$ will be written as R_k with the symmetric and asymmetric parts being denoted by $\sim_{R_k}$ and $\succ_{R_k}$ (or $\sim_k$ and $\succ_k$ for short) respectively.[2] Let B be the set of all complete binary relations on X, and $L \subset B$ the set of all linear orders (complete, transitive and asymmetric binary relations) on X. Lists of individual (strict) preferences (also called profiles) will be written as $u = (L_1^u, L_2^u, ..., L_n^u) \in L^n$ where $L_k^u \in L$ is voter k's preference relation on X in profile u.

A voting rule is a mapping $v : L^n \to B$. The most prominent voting rule is, of course, the simple majority rule. It is defined as a voting rule $f : L^n \to B$ such that for all $u \in L^n$ and all $x, y \in X$, $x \succ_{f(u)} y$ if and only if $\left|\{k \in H : x \succ_k y\}\right| > \left|\{k \in H : y \succ_k x\}\right|$. That is, a candidate x

[1] The algorithms used for the simulations are available from the authors upon request.
[2] Subscripts will be dropped, whenever there is no danger of confusion.

is socially strictly preferred to a candidate *y* if and only if there are more voters preferring *x* over *y* than there are voters preferring *y* over *x*.

Unfortunately, the simple majority rule does not give for every profile of individual preferences a weak order that could be directly used to justify the choice of a particular candidate in the familiar terms of preference optimization. Hence, there have been many attempts to overcome this problem by using social choice rules which yield a weak order for every possible profile of linear rankings, such as scoring rules and in particular Borda's rule. This in turn raises the question of the extent to which desirable properties of simple majority rule are preserved in these alternative procedures. The Condorcet criterion is widely-seen as an obvious criterion for the preservation of such a desirable property.

The Condorcet criterion can be formulated in various ways and strengths. In this paper we follow Nurmi (1988) and Vandercruyssen (1999) in using a weak version of this criterion which only requires that the unique Condorcet winner, if it exists, belongs to the set of winning candidates according to the voting rule under investigation.

A distance measure can now be derived from the position of the Condorcet winner in the weak order resulting from a given scoring rule. For any profile $u \in L^n$ of individual preferences, which has a Condorcet winner, and for every scoring rule *v*, the positional distance with respect to the Condorcet winner is defined as the minimal number of inversions of pairs of adjacent candidates necessary to make the Condorcet winner a (not necessarily unique) element of the winning set of candidates $w_{v(u)} = \left\{ x \in X : x \cdot_{v(u)} y, \forall y \in X \right\}$ according to scoring rule *v*. The value of this distance measure can equivalently be obtained by taking half the Kemeny distance between the weak order determined from the scoring rule, *v(u)*, and the closest weak order that has the Condorcet winner among the top candidates.[3] For any profile $u \in L^n$ with Condorcet winner c_u, voting rule v and winning set $w_{v(u)}$, the positional distance with respect to the Condorcet winner is computed as

$$\Delta_v^u = \frac{1}{2}\Big[\left|\{x \in X : x \succ c_u\}\right| - \left|\{y \in X : c_u \succ y\}\right| - \left|w_{v(u)} \setminus \{c_u\}\right| + (m-1)\Big].$$

Looking at the formula, positional distance with respect to the Condorcet winner c_u increases with the number of candidates that are preferred to c_u and decreases with the number of candidates c_u is preferred to. Moreover, as we use a weak Condorcet criterion, we have to adjust for the fact that the Condorcet winner only has to be among the winning candidates but need not be uniquely top ranked. This will be dealt with by taking into account the cardinality of the winning set. Finally, the term *(m-1)* is introduced to guarantee a zero distance whenever the Condorcet winner is among the winning candidates, i.e. when the Condorcet criterion is satisfied. This, obviously, is a desirable property of any distance-based measure with respect to a certain criterion. Satisfaction of this property in our case in particular guarantees that our distance-based approach is consistent with the probabilistic approach to the measurement of Condorcet efficiency for all cases where the Condorcet criterion is satisfied.

The distance-based measure of the Condorcet efficiency of a given scoring rule is obtained by averaging distances over profiles instead of determining probabilities. For every scoring rule *v*, this measure is obtained as:

$$\Delta_v^{eff} := \frac{\sum_{u \in C} \Delta_v^u}{|C|},$$

where $C \subset L^n$ is the set of profiles for which there exists a Condorcet winner. As can be expected from a distance-based measure, smaller values of Δ_v^{eff} indicate higher Condorcet efficiency.

[3] In general, the Kemeny distance between two binary relations R and R' is twice the number of inversions of pairs of adjacent candidates necessary to obtain the binary relation R' from the binary relation R. For the use of

Our distance-based measure of Condorcet efficiency is closely related to the probabilistic one. For every scoring rule *v*, let $\bar{C}^v \subset C$ denote the set of all profiles such that the Condorcet winner is not in the winning set of scoring rule *v*, i.e. the Condorcet criterion is violated by *v*. Hence the relative distance-based measure $\bar{\Delta}_v := \frac{\sum_{u \in C} \Delta_v^u}{\left|\bar{C}^v\right|}$ shows the extent to which, on average, the Condorcet criterion is violated whenever it is not satisfied.[4] This measure establishes a link between distance-based and probabilistic measures of Condorcet efficiency. Let p_v be the probability that a given voting rule *v* satisfies the Condorcet criterion. Then the distance-based measure of Condorcet efficiency Δ_v^{eff} is obviously related to the probabilistic measure p_v in the following way: $\Delta_v^{eff} = (1 - p_v)\bar{\Delta}_v$.

3 Results

We investigated the Condorcet efficiency of the Borda rule, the plurality rule, and the antiplurality rule by means of computer simulations for various sizes of electorates (5, 15, 25) and numbers of candidates ranging from 3 to 100. We generated 100000 profiles for which there exists a Condorcet winner[5] for every combination of size of the electorate and number of candidates and computed for every scoring rule the corresponding social preference relation. In line with the concept of impartial culture[6] used in the literature on Condorcet efficiency, the individual preferences determining a profile were generated randomly and independently.

As a first result, allowing for larger numbers of candidates than usual in the literature changes the familiar ranking of the scoring rules under consideration with respect to their Condorcet

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the Kemeny distance in social choice theory see e.g. Saari and Merlin (2000).

[4] As positional distance is zero whenever the Condorcet criterion is satisfied, $\sum_{u \in C} \Delta_v^u = \sum_{u \in \bar{C}^v} \Delta_v^u$.

[5] This means that e.g. for 25 voters and 100 candidates almost one million profiles are needed to be generated as the probability of generating profiles with a Condorcet winner decreases with the number of candidates.

[6] For a general discussion on the assumption concerning the distributions of profiles of voter preferences see Berg and Lepelley (1994).

efficiency. This ranking is characterized by a clear superiority of the Borda rule over the plurality rule and the antiplurality rule with the plurality rule performing slightly better than the antiplurality rule. Our simulations as reported in table 1 and in the figures in the appendix, confirm this ranking only for numbers of candidates that are small compared to the size of the electorate. With 5 voters the antiplurality rule has a higher Condorcet efficiency for 10 and more candidates (see table 1), with 15 voters for 33 and more candidates and with 25 voters for 84 and more candidates. With this small number of voters the probabilistic Condorcet efficiency of the antiplurality reaches 99,66% for 100 candidates. In line with the familiar ranking of these three scoring rules, the probabilistic Condorcet efficiency of the plurality rule is constantly worse than that of the Borda rule for any number of candidates. It is also worse than that of the antiplurality rule for any number of candidates larger than some (rather small) level depending on the size of the electorate (see table 1).

n \ m		5		10		15		20		40		60		80		100	
		Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}	Cond eff	Δ_v^{eff}
5	Borda	93,01	0,04	86,25	0,09	83,47	0,12	82,36	0,14	80,44	0,19	79,82	0,22	79,64	0,24	79,53	0,25
	Plur	84,42	0,25	74,63	0,76	70,41	1,53	66,42	2,55	55,24	8,00	48,46	14,51	43,10	21,78	39,50	29,30
	APlur	70,39	0,45	88,68	0,48	93,79	0,42	96,01	0,37	98,58	0,28	99,19	0,24	99,51	0,19	99,66	0,17
15	Borda	89,61	0,06	85,76	0,09	85,38	0,10	85,33	0,10	86,66	0,09	87,78	0,09	88,87	0,08	89,36	0,08
	Plur	73,66	0,34	57,08	1,20	48,96	2,26	44,11	3,49	30,68	9,52	26,36	16,16	24,37	23,38	23,15	31,16
	APlur	62,58	0,47	51,35	1,56	66,14	1,96	75,67	2,04	89,41	1,99	93,54	1,87	95,57	1,73	96,54	1,70
25	Borda	88,39	0,06	85,48	0,09	85,39	0,10	85,71	0,10	87,71	0,08	89,14	0,07	90,17	0,06	91,27	0,06
	Plur	70,27	0,37	52,30	1,30	43,89	2,42	37,60	3,73	27,24	9,70	21,86	16,67	17,74	24,37	15,11	32,30
	APlur	60,46	0,49	43,88	1,64	43,37	2,83	55,11	3,38	77,75	3,99	85,91	3,98	89,76	3,94	92,06	3,86

Table 1: Probabilistic Condorcet efficiency and distance-based Condorcet efficiency for various combinations of numbers of candidates and sizes of electorate

This result is not only of independent interest, it also stresses the significance of a distance-based approach to the measurement of Condorcet efficiency. The figures in the appendix show, for various sizes of electorates, the values of the *relative* distance measure $\overline{\Delta}_v$, which measures the extent to which the Condorcet criterion is violated over all profiles for which a given scoring rule *v* does not satisfy it. These values are similar and increasing in the number of candidates for the plurality rule and the antiplurality rule, while the superiority of the Borda rule is confirmed by constantly low values of this measure. Thus, our distance-based measure of Condorcet efficiency Δ_v^{eff} corrects for the effect of a high number of candidates on the efficiency of the antiplurality rule. For example, with an electorate of 25 voters and 100 candidates, the Condorcet efficiency of the Borda rule is 91,27% and that of the antiplurality rule 92,06%. However, the distance-based measure assigns a value of 0,06 to the Borda rule and 3,86 to the antiplurality rule, indicating a superior distance-based efficiency of the Borda rule (see table 1). Thus, the superiority of the Borda rule extends to larger numbers of candidates than are usual in the literature. However, given consistency of the probabilistic measure of Condorcet efficiency with our distance-based measure, the latter cannot offset the effect of a probabilistic efficiency of almost 100% of the antiplurality rule. This can be seen in the case of an electorate of 5 voters. For this size of the electorate and for more than 64 candidates the antiplurality rule outperforms the Borda rule even with respect to our distance-based measure of Condorcet efficiency (see table 1 and figure 1).

4 Conclusion

The usual probabilistic measurement of Condorcet efficiency favors the antiplurality rule at the expense of the Borda rule for larger numbers of candidates than are usually considered in the literature. A distance-based measure of Condorcet efficiency compensates this effect and extends the superiority of the Borda rule to larger numbers of candidates.

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Appendix

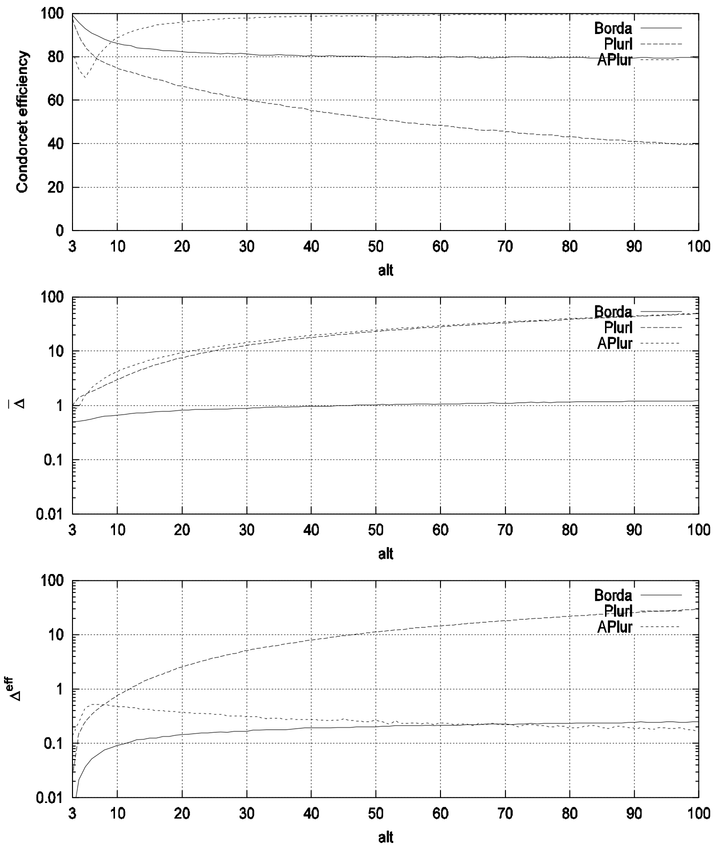

Figure 1: Computations of efficiency measures for a 5-voter electorate

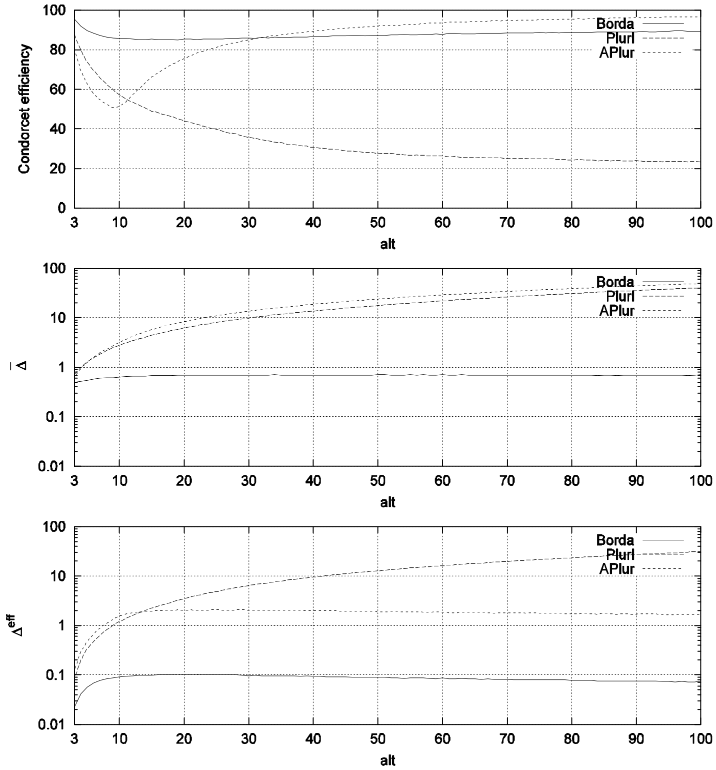

Figure 2: Computations of efficiency measures for a 15-voter electorate

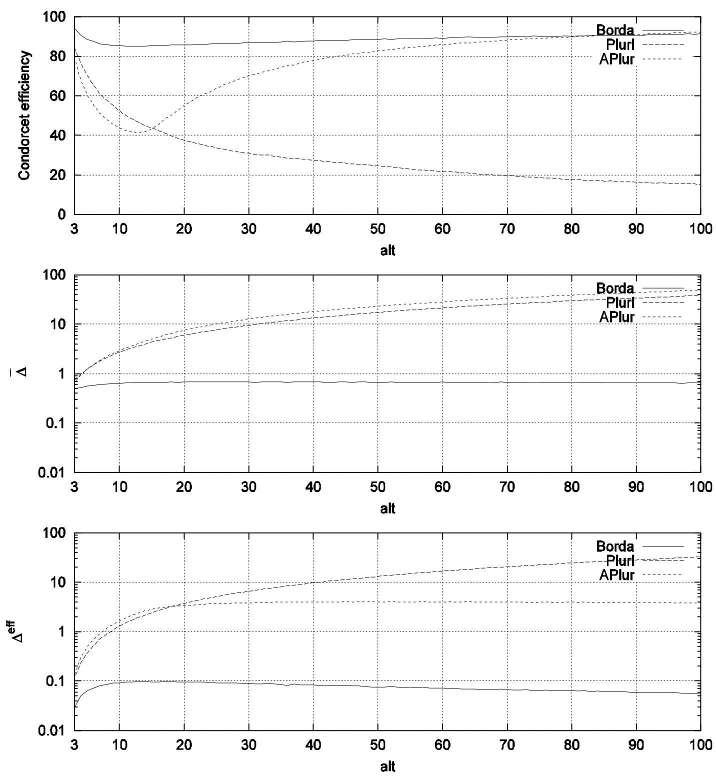

Figure 3: Computations of efficiency measures for a 25-voter electorate