A semi-smooth Newton method for control constrained boundary optimal control of the Navier-Stokes equations

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Abstract

In this paper we study optimal control of the Navier-Stokes equations when the control acts as a pointwise constrained boundary condition of Dirichlet type. The problem is analyzed in the control space $\mathbf{H}_{00}^{1/2}$, the optimality system and second order sufficient optimality conditions are derived. For the numerical solution we apply a semi-smooth Newton method to a regularized version of the original problem and show convergence properties of the method and of the regularized solutions towards the original one.

Key words: Optimal boundary control with control constraints, Navier-Stokes equations, semi-smooth Newton methods

1 Introduction

The Navier-Stokes equations are a widely accepted model for the behaviour of viscous incompressible fluids in the presence of convection. The nonlinear nature of the equations presented, since they were introduced in the XIX$^{\text{th}}$ century, many challenges to obtain existence and uniqueness results, as well as for the development of efficient methods for their numerical solution. Beside many analytical techniques, such as variational solutions (cf. [8,10,18,38]) or semigroups of operators (cf. [14]), many numerical schemes (cf. [15]) and discretization techniques (cf. [19,37,41]) have been applied and, in many cases, developed in the study of the problem.

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The analysis of controlling these equations has a more recent history. It appears that such problems received increased attention since the 1980's. First investigations primarily concentrated on control through body forces, we mention [16] in this respect. After that, distributed control was the subject of many analytical as well as numerical papers for the stationary case (cf. [17,20,34]) and time dependent problems (cf. [1,23,25,29,34]). More recently, optimal control of the Navier-Stokes equations through the action of Dirichlet boundary conditions was analyzed, in [21,22,24,27] for example. In some cases, numerical results, either by solving the optimality system or by optimization methods, were obtained.

The optimal control problem with pointwise control constraints, has received significantly less attention. Since the analysis of the problem yields a variational inequality as optimality condition, the numerical treatment offers new challenges, which were, in general, not explicitly studied. The unique recent reference we know about is [40], where the author applies a semi-smooth Newton method for the solution of the instationary distributed control problem.

In the present paper, we deal with pointwise constrained boundary optimal control of the Navier-Stokes equations. Differently from several previous contributions, where the space $\mathbf{H}_0^1(\Gamma_1)$ is used as control space, we utilize here the space $\mathbf{H}_{00}^{1/2}(\Gamma_1)$ which is the natural space from the variational point of view. The problem is numerically solved by applying a semi-smooth Newton method to a properly regularized problem.

The outline of the paper is as follows. In section 2 we present existence and uniqueness results for the state equations, as well as some regularity results. Section 3 deals with the boundary control problem along the physical boundary. Existence results, first order necessary and a second order sufficient optimality conditions are obtained. In section 4 the semi-smooth Newton method is applied to a regularized version of the original boundary optimal control problem. Convergence results of the algorithm and of the regularized solutions to the original one are proved. Finally in section 5, some selected numerical experiments are presented.

2 State equations

Let us first introduce the notation to be used. We denote by $(\cdot,\cdot)_X$ the inner product in the Hilbert space X and by $\|\cdot\|_X$ the associated norm. The topological dual of X is denoted by X' and the duality pairing is expressed as $\langle\cdot,\cdot\rangle_{X',X}$. For the L^2-inner product and norm no subindices are used. The space of infinitely differentiable functions with compact support in Ω is denoted by $\mathcal{D}(\Omega)$ and its dual, the distributions space, by $\mathcal{D}'(\Omega)$. Here Ω denotes a bounded domain with boundary Γ. The Sobolev space $W^{m,p}(\Omega)$ is the space of $L^p(\Omega)$ functions whose distributional derivatives up to order m are also in $L^p(\Omega)$.

For these spaces a norm is introduced by via

$$\|u\|_{W^{m,p}} = \left(\sum_{[j]\le m} \left\| D^j u \right\|_{L^p}^p \right)^{1/p},$$

where D^j denotes the differentiation operator with respect to the multi-index $j = (j_1, ... j_n)$, i.e. $D^j = \frac{\partial^{[j]}}{\partial x^{j_1} \cdots \partial x^{j_n}}$, with $[j] = \sum_{i=1}^n j_i$. If $p = 2$ we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, which is a Hilbert space with the scalar product

$$(u, v)_{H^m} = \sum_{[j]\le m} (D^j u, D^j v).$$

The closure of $\mathcal{D}(\Omega)$ in the $W^{m,p}(\Omega)$ norm is denoted by $W_0^{m,p}(\Omega)$. If Ω is smooth enough, then $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For the space $H_0^1(\Omega)$ the Poincaré inequality holds, i.e.

$$\|u\| \le c_0 \|\nabla u\|, \text{ for all } u \in H_0^1(\Omega),$$

where c_0 is a constant dependent of Ω. Thus, in $H_0^1(\Omega)$ the H^1-norm is equivalent to the norm

$$\|u\|_{H_0^1} = \|\nabla u\|$$

and $H_0^1(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H_0^1} = (\nabla u, \nabla v).$$

The dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. Since m-dimensional vector-functions will be frequently used, we introduce the bold notation for the product-spaces, for example $\mathbf{L}^2(\Omega) = \prod_{i=1}^m L^2(\Omega)$, and provide them with the Euclidean product norm. The divergence free distributions space is denoted by $\mathcal{V}$ and its closure in $\mathbf{H}_0^1(\Omega)$ by $\mathbf{V}$, which can be also characterized as $\mathbf{V} = \{v \in \mathbf{H}_0^1(\Omega) : \text{div } v = 0\}$. Additionally, let $\mathbf{H}_0^{1/2} = \{v \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma v \cdot \vec{n} \; d\Gamma = 0\}$ and $\mathbf{H} = \{v \in \mathbf{H}^1(\Omega) : \text{div } v = 0\}$ be subspaces of $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^1(\Omega)$ respectively. The functional $T(u) = \int_\Gamma u \cdot \vec{n} \, d\Gamma$ is linear and bounded from $\mathbf{L}^2(\Gamma) \to \mathbb{R}$ and, due to the embedding $\mathbf{H}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}^2(\Gamma)$ with continuous injection, it is also continuous from $\mathbf{H}^{1/2}(\Gamma) \to \mathbb{R}$. Hence $\mathbf{H}_0^{1/2} = \ker(T)$ is a closed linear subspace and consequently a Hilbert space with the scalar product induced by $\mathbf{H}^{1/2}(\Gamma)$. By the same arguments we can argue that the divergence operator is a bounded linear operator from $\mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Omega)$. Consequently $\mathbf{H}$ is a Hilbert space with the $\mathbf{H}^1(\Omega)$ norm.

Henceforth we let Ω denote a bounded domain in $\mathbb{R}^2$. The stationary Navier-Stokes equations are given by

$$-\nu \Delta y + (y \cdot \nabla) y + \nabla p = f \tag{1}$$

$$\text{div } y = 0 \tag{2}$$

$$y|_\Gamma = g, \tag{3}$$

where $f \in \mathbf{H}^{-1}(\Omega),\ g \in \mathbf{H}_0^{1/2}$ and $(y \cdot \nabla) y = \left(y_1 \frac{\partial y_1}{\partial x_1} + y_2 \frac{\partial y_1}{\partial x_2},\ y_1 \frac{\partial y_2}{\partial x_1} + y_2 \frac{\partial y_2}{\partial x_2} \right)$.

Define the trilinear form $c : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ by $c(u, v, w) = ((u \cdot \nabla)v, w)$. Multiplying (1) by test-functions $v \in \mathcal{V}$, a weak formulation of the Navier-Stokes equations, is given by: find $y \in \mathbf{H}$ such that

$$a(y, v) + c(y, y, v) = \langle f, v\rangle_{\mathbf{V}',\mathbf{V}}, \text{ for all } v \in \mathbf{V} \tag{4}$$

$$\gamma_0 y = g, \tag{5}$$

where γ_0 stands for the trace operator.

Conversely, if $y \in \mathbf{H}$ satisfies (4), then

$$\langle -\nu\Delta y + (y \cdot \nabla)y - f, v\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)} = 0, \quad \text{for all } v \in \mathcal{V}$$

and, consequently (see [10], pg. 8), there exists a distribution $p \in L^2_0(\Omega)$ such that (1) is satisfied in the distributional sense. Equations (2) and (3) are satisfied in a distributional and trace theorem sense, respectively. The following result is well-known from the literature, see e.g. [8,18,38].

Theorem 1 *Let Ω be an open bounded set of $\mathbb{R}^2$. The trilinear form c is continuous on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and satisfies:*

(1) $c(u, v, v) = 0$ for all $u \in \mathbf{H}$ with $\gamma_n u = 0$, for all $v \in \mathbf{H}^1(\Omega)$.
(2) $c(u, v, w) = -c(u, w, v)$ for all $u \in \mathbf{H}$ with $\gamma_n u = 0$, for all $v, w \in \mathbf{H}^1(\Omega)$.
(3) $c(u, v, w) = -c(u, w, v)$ for all $u \in \mathbf{H}$, $v \in \mathbf{H}^1(\Omega)$, $w \in \mathbf{V}$.
(4) $c(u, v, w) = ((\nabla v)^T w, u)$

Corollary 2 *Let Ω be an open bounded set of $\mathbb{R}^2$. The form c is continuous on $\mathbf{H}^1_0(\Omega) \times \mathbf{H}^1_0(\Omega) \times \mathbf{H}^1_0(\Omega)$.*

Proof. Follows using the same steps as in the proof of Theorem 1. □

Theorem 3 *Let Ω be an open bounded subset of $\mathbb{R}^2$ and let $f \in \mathbf{H}^{-1}(\Omega)$. Then, problem (4), (5), with homogeneous boundary conditions, has at least one variational solution $y \in \mathbf{V}$ and there exists a distribution $p \in L^2_0(\Omega)$ such that (1), (2) and (3) are satisfied. Moreover, the solution satisfies the following estimate:*

$$\|y\|_{\mathbf{V}} \leq \frac{1}{\nu} \|f\|_{\mathbf{V}'} . \tag{6}$$

Proof. For a detailed proof see [8,18,38]. □

Theorem 4 *If $\nu^2 > \mathcal{N} \|f\|_{\mathbf{V}'}$, where $\mathcal{N} = \sup\limits_{u,v,w\in\mathbf{V}} \frac{|c(u,v,w)|}{\|u\|_{\mathbf{V}}\|v\|_{\mathbf{V}}\|w\|_{\mathbf{V}}}$, then the solution for (4), (5), with homogeneous boundary conditions, is unique.*

Proof. We refer to [8,18,38]. □

Lemma 5 *For every $\varepsilon > 0$, there exists a function $\hat{y} \in \mathbf{H}^1(\Omega)$ such that div $\hat{y} = 0$, $\gamma_0 \hat{y} = g$ and*

$$|c(v, \hat{y}, v)| \leq \varepsilon \|v\|_{\mathbf{V}}^2 \text{ for all } v \in \mathbf{V}.$$

Proof. A detailed proof can be found in [38], pg. 175. □

Theorem 6 *Let $f \in \mathbf{H}^{-1}(\Omega)$ and $g \in \mathbf{H}_0^{1/2}$. Then, there exists at least one solution for the non-homogeneous problem (1), (2), (3).*

Proof. For the proof see [8,18,38]. □

Theorem 7 *If $\|\hat{y}\|_{\mathbf{H}}$ is sufficiently small, so that*

$$|c(v, \hat{y}, v)| \leq \frac{\nu}{2} \|v\|_{\mathbf{V}}^2 \text{ for all } v \in \mathbf{V}$$

and ν satisfies $\nu^2 > 4\mathcal{N} \|F\|_{\mathbf{V}'}$, with $F = f + \nu\Delta\hat{y} - (\hat{y} \cdot \nabla)\hat{y}$, then there exists a unique solution (y, p) for the problem (1), (2), (3). Additionally the following estimate holds

$$\|y - \hat{y}\|_{\mathbf{V}} \leq \frac{2}{\nu} \|F\|_{\mathbf{V}'} . \tag{7}$$

Proof. We refer the reader to [38]. □

As for the Stokes case, extra regularity of the solution can be obtained if the right hand side and the boundary condition are smooth enough.

Theorem 8 *Let $f \in \mathbf{L}^2(\Omega)$ and $g \in \mathbf{H}_0^{1/2} \cap \mathbf{H}^{3/2}(\Gamma)$. Then $y \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$.*

Proof. We refer to [11]. □

3 Optimal control problem

In this section we consider optimal control problem of the Navier-Stokes equations, when the control is allowed to be exerted on part of the boundary, under pointwise constraints. The class of admissible controls will be a subset of the Hilbert space $\mathbf{H}_{00}^{1/2}(\Gamma_1)$.

3.1 Control space

We begin with the definition of the trace space

$$\mathbf{H}^{1/2}_{00}(\Gamma_1) = \{v \in \mathbf{L}^2(\Gamma_1) : \text{ there exists } w \in \mathbf{H}^1(\Omega), w|_{\Gamma_2} = 0, w|_{\Gamma_1} = v\}. \tag{8}$$

It can be verified that $\mathbf{H}^{1/2}_{00}(\Gamma_1)$ is a closed subspace of $\mathbf{H}^{1/2}(\Gamma_1)$ (the restriction of the elements of $\mathbf{H}^{1/2}(\Gamma)$ to Γ_1), which can also be defined as

$$\mathbf{H}^{1/2}_{00}(\Gamma_1) = \{v \in \mathbf{L}^2(\Gamma_1) : \text{ there exists } g \in \mathbf{H}^{1/2}(\Gamma), g|_{\Gamma_2} = 0, g|_{\Gamma_1} = v\}. \tag{9}$$

It can be verified (cf. [9], pg. 397) that $\mathbf{H}^{1/2}_{00}(\Gamma_1)$ is a Hilbert space endowed with the scalar product

$$\begin{aligned}(u, v)_{\mathbf{H}^{1/2}_{00}} &= (u, v)_{\mathbf{L}^2(\Gamma_1)} + ((-\Delta_\Gamma)^{1/4} u, (-\Delta_\Gamma)^{1/4} u)_{\mathbf{L}^2(\Gamma_1)} \\ &= (u, v)_{\mathbf{L}^2(\Gamma_1)} + ((-\Delta_\Gamma)^{1/2} u, u)_{\mathbf{L}^2(\Gamma_1)}\end{aligned}$$

and that the following continuous and dense injections hold:

$$\mathbf{H}^1_0(\Gamma_1) \hookrightarrow \mathbf{H}^{1/2}_{00}(\Gamma_1) \hookrightarrow \mathbf{L}^2(\Gamma_1) \hookrightarrow (\mathbf{H}^{1/2}_{00}(\Gamma_1))' \hookrightarrow \mathbf{H}^{-1}(\Gamma_1) \tag{10}$$

Our control space is defined as

$$\mathcal{U} = \{v \in \mathbf{H}^{1/2}_{00}(\Gamma_1) : \int_{\Gamma_1} v \cdot \vec{n}\, d\Gamma = 0\}. \tag{11}$$

Due to the linearity and continuity of $T(u) = \int_{\Gamma_1} v \cdot \vec{n}\, d\Gamma$ from $\mathbf{L}^2(\Gamma_1) \to \mathbb{R}$, together with the continuous embedding $\mathbf{H}^{1/2}_{00}(\Gamma_1) \hookrightarrow \mathbf{L}^2(\Gamma_1)$, we conclude that $\mathcal{U}$ is itself a Hilbert space with the induced norm. Let $\mathcal{U}^- = \{f \in \mathbf{H}^{-1/2}_{00}(\Gamma_1) : \langle f, v\rangle_{\mathbf{H}^{-1/2}_{00}, \mathbf{H}^{1/2}_{00}} \leq 0, \text{ for all } v \in \mathcal{U}\}$ be the negative polar cone of $\mathcal{U}$. It is simple to verify that $\mathcal{U}^- = \{\sigma\vec{n} : \sigma \in \mathbb{R}\}$.

3.2 Problem statement

Let Γ_1 be an open connected subset of Γ. The problem we are concerned with can be stated as follows: given $f \in \mathbf{L}^2(\Omega)$, $z_d \in \mathbf{H}^1(\Omega)$ and $\alpha > 0$, find $(y^*, u^*) \in \mathbf{H} \times \mathcal{U}_{ad}$, with $\mathcal{U}_{ad} = \{v \in \mathcal{U} : v \leq b \text{ a.e.}\}$, which solves:

$$\begin{cases} \min\limits_{(y,u)\in \mathbf{H}\times\mathcal{U}_{ad}} \quad J(y,u) = \frac{1}{2}\|y - z_d\|^2_{\mathbf{H}} + \frac{\alpha}{2}\|u\|^2_{\mathbf{L}^2(\Gamma_1)} \\ \text{subject to:} \\ a(y,v) + c(y,y,v) = (f,v), \text{ for all } v \in \mathbf{V} \\ \gamma_0 y = g + \mathcal{B}u, \end{cases} \tag{$\mathcal{P}$}$$

where $b \in \mathbf{H}_{00}^{1/2}(\Gamma_1)$, $g \in \mathbf{H}_0^{1/2}$ and $\mathcal{B} \in \mathcal{L}(\mathbf{H}_{00}^{1/2}(\Gamma_1), \mathbf{H}^{1/2}(\Gamma))$ is defined by $\mathcal{B}u = \begin{cases} u & \text{in } \Gamma_1 \\ 0 & \text{in } \Gamma\backslash\Gamma_1 \end{cases}$.

To describe the constraints in $(\mathcal{P})$ we define the constraint operator $G : \mathbf{H} \times \mathcal{U}_{ad} \longrightarrow \mathbf{V}' \times \mathbf{H}_0^{1/2}$ by

$$G(y,u) = \begin{pmatrix} a(y,\cdot) + c(y,y,\cdot) - (f,\cdot) \\ \gamma_0 y - g - \mathcal{B}u \end{pmatrix}$$

and formulate the restrictions as $G(y,u) = 0$ in $\mathbf{V}' \times \mathbf{H}_0^{1/2}$.

It can be seen that G is Fréchet differentiable with the derivative, evaluated at (y,u) in the direction (w,h), given by:

$$G'(y,u)(w,h) = \begin{pmatrix} a(w,\cdot) + c(y,w,\cdot) + c(w,y,\cdot) \\ \gamma_0 w - \mathcal{B}h \end{pmatrix}$$

3.3 Existence

Let us assume that $\mathcal{U}_{ad} \neq \emptyset$ and define $\mathcal{T}_{ad} = \{(y,u) \in \mathbf{H} \times \mathcal{U}_{ad} : G(y,u) = 0\}$.

Theorem 9 *There exists an optimal solution for* $(\mathcal{P})$*.*

Proof. Since there is some $\hat{u} \in \mathcal{U}_{ad}$, we know from the existence theorem of the non-homogeneous Navier-Stokes equations that there exists a $y \in \mathbf{H}$ which satisfies:

$$\begin{aligned} a(y,v) + c(y,y,v) &= (f,v) \text{ for all } v \in \mathbf{V} \\ \gamma_0 y &= g + \mathcal{B}\hat{u}. \end{aligned}$$

Additionally $\mathcal{T}_{ad} \neq \emptyset$ and $J(y,\hat{u}) < \infty$.

Let $\{(y_k,u_k)\}$ be a minimizing sequence in $\mathcal{T}_{ad}$. Due to the definition of $\mathcal{T}_{ad}$ and the fact that $J(y_k,u_k)$ tends to the infimum, $J(y_k,u_k) \leq C$ with C independent of k. Also,

$$J(y_k,u_k) \geq \|y_k - z_d\|_{\mathbf{H}}^2 \geq \frac{1}{2}\|y_k\|_{\mathbf{H}}^2 - \|z_d\|_{\mathbf{H}}^2,$$

and consequently $\{\|y_k\|_{\mathbf{H}}\}_{k=1}^{\infty}$ is uniformly bounded.

Additionally, due to the definition of $\mathbf{H}^{1/2}_{00}(\Gamma_1)$ and the trace theorem,

$$\begin{aligned}\|u_k\|_{\mathcal{U}} &= \|\mathcal{B}u_k\|_{\mathbf{H}^{1/2}(\Gamma)} = \|\gamma_0 y_k - g\|_{\mathbf{H}^{1/2}(\Gamma)} \\ &\leq \|\gamma_0 y_k\|_{\mathbf{H}^{1/2}(\Gamma)} + \|g\|_{\mathbf{H}^{1/2}(\Gamma)} \leq C_3 \|y_k\|_{\mathbf{H}} + \|g\|_{\mathbf{H}^{1/2}(\Gamma)} < \infty.\end{aligned}$$

Thus, $\|u_k\|_{\mathcal{U}}$ is bounded and we may extract a weakly convergent subsequence $\{(y_k, u_k)\}$ such that $y_k \rightharpoonup y^*$ in $\mathbf{H}$ and $u_k \rightharpoonup u^*$ in $\mathcal{U}$.

The set $\{v \in \mathcal{U} : v \leq b\}$ is closed and convex in $\mathbf{H}^{1/2}_{00}(\Gamma_1)$ and hence $u^* \leq b$.

Additionally (y_k, u_k) satisfies the system

$$\begin{aligned}a(y_k, v) + c(y_k, y_k, v) &= (f, v) \text{ for all } v \in \mathbf{V} \\ \gamma_0 y_k &= g + \mathcal{B}u_k.\end{aligned}$$

In order to see that (y^*, u^*) is solution of the Navier-Stokes equations, the only problem is to pass to the limit in the nonlinear form $c(y_n, y_n, v)$. Due to the weakly sequentially continuity of $c(\cdot,\cdot,\cdot)$ (cf. [18], pg. 286), it follows that $c(y_n, y_n, v) \to c(y^*, y^*, v)$. Hence, due also to the linearity and continuity of $a(\cdot,\cdot)$ and the trace operator, $(y^*, u^*) \in \mathcal{T}_{ad}$.

The functional $J(y,u)$ is weakly lower semi-continuous and it follows that

$$J(y^*, u^*) = \inf_{(y,u)\in\mathcal{T}_{ad}} J(y,u). \qquad \square$$

3.4 First order necessary conditions

In this subsection we establish a condition for a pair (y, u) to satisfy the regular point condition (cf. [42], pg. 50). Thereafter the existence of Lagrange multipliers is shown.

Lemma 10 *Let (y^*, u^*) be a feasible pair. If $\nu > \mathcal{M}(y^*)$, with $\mathcal{M}(y) = \sup_{v\in\mathbf{V}} \frac{|c(v,y,v)|}{\|v\|^2_{\mathbf{V}}}$, then (y^*, u^*) satisfies the regular point condition.*

Proof. Given $(d, e) \in \mathbf{V}' \times \mathbf{H}^{1/2}_0$ it suffices to show the existence of $(w, h) \in \mathbf{H} \times \mathcal{U}_{ad}$ and $\theta \geq 0$ such that:

$$\begin{aligned}a(w, v) + c(y^*, w, v) + c(w, y^*, v) &= \langle d, v\rangle_{\mathbf{V}',\mathbf{V}} \\ \gamma_0 w &= e + \theta\mathcal{B}(h - u^*).\end{aligned}$$

We set $\theta = 1,\ h = u^*$ and utilizing Lemma 5, there exists $\bar{w}$ such that $\gamma_0 \bar{w} = e,\ \operatorname{div} \bar{w} = 0$. Setting $\hat{w} = w - \bar{w}$ the problem consists in finding $\hat{w} \in \mathbf{V}$ such that:

$$\begin{aligned} a_1(\hat{w}, v) := a(\hat{w}, v) + c(y^*, \hat{w}, v) + c(\hat{w}, y^*, v) = \langle d, v \rangle \\ - a(\bar{w}, v) - c(y^*, \bar{w}, v) - c(\bar{w}, y^*, v) =: \langle F, v \rangle \quad \text{for all } v \in \mathbf{V}. \end{aligned} \tag{12}$$

Clearly $a_1(\cdot, \cdot)$ is bilinear. Below we verify continuity and coercivity of this form. Existence of $\hat{w} \in \mathbf{V}$ then follows from the Lax-Milgram theorem.

Continuity of the bilinear form a_1 follows from the properties of the forms a and c in the following way:

$$\begin{aligned} |a_1(w, v)| &= |a(w, v) + c(y^*, w, v) + c(w, y^*, v)| \leq |a(w, v)| + |c(y^*, w, v)| + |c(w, y^*, v)| \\ &\leq \nu \|w\|_{\mathbf{V}} \|v\|_{\mathbf{V}} + \mathcal{N} \|y^*\|_{\mathbf{H}} \|w\|_{\mathbf{V}} \|v\|_{\mathbf{V}} + \mathcal{N} \|w\|_{\mathbf{V}} \|y^*\|_{\mathbf{H}} \|v\|_{\mathbf{V}} \\ &= (\nu + 2\mathcal{N} \|y^*\|_{\mathbf{H}}) \|w\|_{\mathbf{V}} \|v\|_{\mathbf{V}}. \end{aligned}$$

The coercivity is obtained next:

$$\begin{aligned} |a_1(v, v)| &= |a(v, v) + c(y^*, v, v) + c(v, y^*, v)| = |a(v, v) - c(v, v, y^*)| \\ &\geq |a(v, v)| - |c(v, v, y^*)| \geq \nu \|v\|_{\mathbf{V}}^2 - \mathcal{M}(y^*) \|v\|_{\mathbf{V}}^2 \\ &= (\nu - \mathcal{M}(y^*)) \|v\|_{\mathbf{V}}^2, \end{aligned}$$

and, since $\nu > \mathcal{M}(y^*)$ by hypothesis, existence of $\hat{w}$ follows. □

Remark 11 If (y^*, u^*) satisfies the hypothesis of Theorem 7, then it also satisfies the regular point condition. In fact, due to the first hypothesis of Theorem 7 and the boundedness of the nonlinear form we obtain that

$$\begin{aligned} |c(v, y^*, v)| &= |c(v, y^* - \hat{y}, v) + c(v, \hat{y}, v)| \leq |c(v, y^* - \hat{y}, v)| + \frac{\nu}{2} \|v\|_{\mathbf{V}}^2 \\ &\leq \mathcal{N} \|y^* - \hat{y}\|_{\mathbf{V}} \|v\|_{\mathbf{V}}^2 + \frac{\nu}{2} \|v\|_{\mathbf{V}}^2. \end{aligned}$$

Using $F = f + \nu \Delta \hat{y} - (\hat{y} \cdot \nabla)\hat{y}$, we get from Theorem 7 that $\|y^* - \hat{y}\|_{\mathbf{V}} \leq \frac{2}{\nu} \|F\|_{\mathbf{V}'}$, which implies

$$|c(v, y^*, v)| \leq \frac{2}{\nu} \mathcal{N} \|F\|_{\mathbf{V}'} \|v\|_{\mathbf{V}}^2 + \frac{\nu}{2} \|v\|_{\mathbf{V}}^2.$$

Since by hypothesis $\nu^2 > 4\mathcal{N} \|F\|_{\mathbf{V}'}$, we obtain that

$$|c(v, y^*, v)| < \nu \|v\|_{\mathbf{V}}^2.$$

Hence the hypothesis of Lemma 10 is satisfied.

Theorem 12 *Let (y^*, u^*) be a local optimal solution for the control problem $(\mathcal{P})$, with $\nu > \mathcal{M}(y^*)$. Then, there exist Lagrange multipliers $(\lambda, \xi) \in \mathbf{V} \times \mathbf{H}_0^{-1/2}$ such that for all $(w, h) \in$*

$\mathbf{H} \times \mathcal{C}(u^*)$

$$(y^* - z_d, w)_{\mathbf{H}} + (\alpha u^*, h) + a(\lambda, w) + c(y^*, w, \lambda) \\ + c(w, y^*, \lambda) + \langle \xi, \gamma_0 w - \mathcal{B}h \rangle_{\mathbf{H}_0^{-1/2}, \mathbf{H}_0^{1/2}} \geq 0, \quad (13)$$

where $\mathcal{C}(u^*) = \{\theta(v - u^*),\ v \in \mathcal{U}_{ad},\ \theta \geq 0\}$.

Proof. From the hypothesis, (y^*, u^*) satisfies the regular point condition and, consequently, there exist multipliers $(\lambda, \xi) \in \mathbf{V} \times \mathbf{H}_0^{-1/2}$ such that:

$$J'(y^*, u^*)(w, h) + \langle (\lambda, \xi), G'(y^*, u^*)(w, h) \rangle \geq 0 \text{ for all } (w, h) \in \mathbf{H} \times \mathcal{C}(u^*).$$

The theorem follows from

$$\langle (\lambda, \xi), G'(y^*, u^*)(w, h) \rangle = a(\lambda, w) + c(y^*, w, \lambda) \\ + c(w, y^*, \lambda) + \langle \xi, \gamma_0 w - \mathcal{B}h \rangle_{\mathbf{H}_0^{-1/2}, \mathbf{H}_0^{1/2}}. \qquad \square$$

3.5 Optimality system

In this subsection we derive an optimality system from (13).

Lemma 13 *Let* $\eta \in \mathbf{H}_{00}^{-1/2}$ *satisfy* $\langle \eta, h \rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} \leq 0$, *for all* $h \in \mathcal{C}(u^*)$. *Then there exist* $\mu \in \mathbf{H}_{00}^{-1/2}$ *and* $\sigma \in \mathbb{R}$ *such that* $\mu = \eta + \sigma \vec{n}$ *and* $\langle \mu, h \rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} \leq 0$, *for all* $h \in \mathcal{K}(u^*)$, *with* $\mathcal{K}(u^*) = \{\theta(v - u^*) : v \in \mathbf{H}_{00}^{1/2}(\Gamma_1),\ v \leq b,\ \theta \geq 0\}$.

Proof. It is easy to verify that $\mathcal{C}(u^*) = \mathcal{K}(u^*) \cap \mathcal{U}$. From convex analysis (cf. [6], pg. 32) we get that for closed convex cones K_1, K_2 in a Hilbert space X, $(K_1 \cap K_2)^- = \overline{\{K_1^- + K_2^-\}}$, where $K^- = \{f \in X' : \langle f, x \rangle_{X', X} \leq 0, \text{ for all } x \in K\}$. Applying this result to $\mathcal{K}(u^*)$ and $\mathcal{U}$, and observing that $\mathcal{K}(u^*)^- + \mathcal{U}^-$ is closed we obtain that $\mathcal{C}(u^*)^- = \mathcal{K}(u^*)^- + \mathcal{U}^-$. Due to the characterization of $\mathcal{U}^-$ obtained in Section 3.1 the result follows. $\square$

Theorem 14 *Let* (y^*, u^*) *be an optimal solution for the control problem* $(\mathcal{P})$, *which satisfies* $\nu > \mathcal{M}(y^*)$. *If* $f \in \mathbf{L}^2(\Omega)$ *and* $z_d \in \mathbf{H}^2(\Omega)$, *then* (y^*, u^*) *satisfies, together with* $(\lambda, \xi) \in \mathbf{V} \times \mathbf{H}_0^{-1/2}$, *the following optimality system:*

$$\begin{cases} a(y^*, v) + c(y^*, y^*, v) = (f, v) \text{ for all } v \in \mathbf{V} \\ \gamma_0 y = g + \mathcal{B}u^* \\ a(\lambda, w) + c(y, w, \lambda) + c(w, y, \lambda) + \langle \xi, \gamma_0 w \rangle_{\mathbf{H}_0^{-1/2}, \mathbf{H}_0^{1/2}} = (z_d - y^*, w)_{\mathbf{H}}, \text{ for all } w \in \mathbf{H} \\ (\alpha u^*, h) - \langle \mathcal{B}^* \xi, h \rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} \geq 0 \text{ for all } h \in \mathcal{C}(u^*), \end{cases}$$

which corresponds to the variational formulation of:

$$\begin{cases} -\nu\Delta y^* + (y^* \cdot \nabla)y^* + \nabla p = f \\ div\ y = 0 \\ y|_\Gamma = g + \mathcal{B}u^* \\ -\nu\Delta\lambda - (y^* \cdot \nabla)\lambda + (\nabla y^*)^T\lambda + \nabla\phi = (I-\Delta)(z_d - y^*) \\ div\ \lambda = 0 \\ \lambda|_\Gamma = 0 \\ \mu = \mathcal{B}^*(\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}) - \alpha u^* + \sigma\vec{n} \quad in\ \mathbf{H}_{00}^{1/2}(\Gamma_1) \\ u^* \leq b \\ \langle \mu, v - b\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}} \leq 0,\ for\ all\ v \in \mathbf{H}_{00}^{1/2}(\Gamma_1),\ v \leq b \\ \langle \mu, u^* - b\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}} = 0, \end{cases} \tag{14}$$

where $\sigma \in \mathbb{R}$ and $p, \phi \in L_0^2(\Omega)$ denote the pressure and adjoint pressure respectively.

Proof. From the necessary condition we obtain, taking $h = 0$, that for all $w \in \mathbf{H}$,

$$a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) + \langle \xi, \gamma_0 w\rangle_{\mathbf{H}_0^{-1/2},\mathbf{H}_0^{1/2}} = (z_d - y, w)_{\mathbf{H}}.$$

If, additionally, we take the test functions in $\mathbf{V}$,

$$a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) = (\nabla(z_d - y^*), \nabla w) + (z - y^*, w) \quad \text{for all } w \in \mathbf{V},$$

which corresponds to the variational formulation of:

$$\begin{aligned} -\nu\Delta\lambda - (y^* \cdot \nabla)\lambda + (\nabla y^*)^T\lambda + \nabla\phi &= (I-\Delta)(z_d - y^*) \\ \text{div } \lambda &= 0 \\ \lambda|_\Gamma &= 0. \end{aligned}$$

Considering again $\mathbf{H}$ as test functions space and applying integration by parts, we obtain:

$$\begin{aligned} -\nu(\Delta\lambda, w) + \langle \nu\frac{\partial\lambda}{\partial n}, \gamma_0 w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} + c(y^*, w, \lambda) + c(w, y^*, \lambda) \\ + \langle \xi, \gamma_0 w\rangle_{\mathbf{H}_0^{-1/2},\mathbf{H}_0^{1/2}} = (\nabla(z_d - y^*), \nabla w) + (z_d - y^*, w) \quad \text{for all } w \in \mathbf{H} \end{aligned}$$

$$= (z_d - y^* - \Delta(z_d - y^*), w) + \langle \frac{\partial}{\partial n}(z_d - y^*), \gamma_0 w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} \quad \text{for all } w \in \mathbf{H},$$

which implies that

$$\langle \frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}, \gamma_0 w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} = \langle \xi, \gamma_0 w\rangle_{\mathbf{H}_0^{-1/2},\mathbf{H}_0^{1/2}} \quad \text{for all } w \in \mathbf{H}$$

and, hence,

$$\xi = \frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n} \text{ in } \mathbf{H}_0^{-1/2}. \tag{15}$$

The term on the right hand side of (15) is well-defined in $\mathbf{H}^{-1/2}(\Gamma)$ (see Theorem 15 below).

Taking $w = 0$ in the necessary condition and replacing ξ yields:

$$(\alpha u^*, h) - \langle \mathcal{B}^*(\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}), h\rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} \geq 0 \text{ for all } h \in \mathcal{C}(u^*).$$

From Lemma 13 we get the existence of $\mu \in \mathbf{H}_{00}^{-1/2}(\Gamma_1)$ and $\sigma \in \mathbb{R}$ such that $\mu = \mathcal{B}^*(\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}) - \alpha u^* + \sigma\vec{n}$ and the following complementarity problem holds:

$$\begin{aligned}
&\langle \mu, v - b\rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} \leq 0 \text{ for all } v \in \mathbf{H}_{00}^{1/2}(\Gamma_1), \; v \leq b \text{ a.e.,} \\
&\langle \mu, u^* - b\rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}} = 0. \qquad \square
\end{aligned}$$

Theorem 15 *If the conditions of Theorem 14 are satisfied, then* $\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}$ *belongs to* $\mathbf{H}^{-1/2}(\Gamma)$.

Proof. First we will show that λ has some extra regularity. The variational formulation of the adjoint equations can also be written as:

$$\begin{aligned}
a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) = (\nabla z_d, \nabla w) + \\
(z - y^*, w) + \frac{1}{\nu}c(y^*, y^*, w) - \frac{1}{\nu}(f, w) \text{ for all } w \in \mathbf{V},
\end{aligned}$$

which corresponds to the weak formulation of:

$$-\nu\Delta\lambda + \nabla\Psi = z_d - y^* - \Delta z_d + (y^* \cdot \nabla)\lambda - (\nabla y^*)^T\lambda + \frac{1}{\nu}(y^* \cdot \nabla)y^* - \frac{1}{\nu}f \tag{16}$$

$$\text{div } \lambda = 0 \tag{17}$$

$$\lambda|_\Gamma = 0, \tag{18}$$

with $\Psi \in L_0^2(\Omega)$.

Proceeding as in the proof of Theorem 8 we obtain that $(y^* \cdot \nabla)y^* \in \mathbf{W}^{-1,\alpha}(\Omega)$ and $(y^* \cdot \nabla)\lambda$, $(\nabla y^*)^T\lambda \in \mathbf{W}^{-1,\alpha}(\Omega)$, for all $\alpha > 2$. Since by assumption $z_d \in \mathbf{H}^2(\Omega)$, the right hand side of (16) belongs to $\mathbf{W}^{-1,\alpha}(\Omega)$. Applying the regularity results for the Stokes equations (cf. [38], pg. 23), we obtain that $\lambda \in \mathbf{W}^{1,\alpha}(\Omega)$. Due to the embedding $\mathbf{W}^{1,\alpha}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ we obtain that $(\nabla y^*)^T\lambda \in \mathbf{L}^2(\Omega)$. Since $y^* \in \mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$, for all $\beta \in [1, \infty)$ and $\frac{\partial \lambda_i}{\partial x_j} \in L^\alpha(\Omega)$, for $i, j = 1, 2$ and all $\alpha > 2$, we get, taking for example $\alpha = \beta = 4$, that $(y^* \cdot \nabla)\lambda \in \mathbf{L}^2(\Omega)$. Hence

$$z_d - y^* + (y^* \cdot \nabla)\lambda - (\nabla y^*)^T\lambda \in \mathbf{L}^2(\Omega).$$

To prove the claim we express $\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}$ as $(\nabla(-\nu\lambda + z_d - y^*) + \phi I) \cdot \vec{n}$, with I the identity matrix. Next, we verify that

$$\begin{aligned}
\text{div } (\nabla(-\nu\lambda_1 + z_{d_1} - y_1^*) + \phi(1,0)^T) &= -\nu\Delta\lambda_1 + \Delta z_{d_1} - \Delta y_1^* + \frac{\partial\phi}{\partial x_1} \\
\text{div } (\nabla(-\nu\lambda_2 + z_{d_2} - y_2^*) + \phi(0,1)^T) &= -\nu\Delta\lambda_2 + \Delta z_{d_2} - \Delta y_2^* + \frac{\partial\phi}{\partial x_2},
\end{aligned}$$

which, due to the adjoint equations, are equal to the components of $z_d - y^* + (y^* \cdot \nabla)\lambda - (\nabla y^*)^T\lambda \in \mathbf{L}^2(\Omega)$. Hence (cf. [38], pg. 7), $\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n} \in \mathbf{H}^{-1/2}(\Gamma)$. □

3.6 Second order sufficient condition

In this subsection we present sufficient conditions for a pair (y^*, u^*), satisfying $\nu > \mathcal{M}(y^*)$, to be locally, respectively globally, optimal. These conditions will be satisfied if the controlled state is close enough to the desired one.

For the analysis let us introduce the Lagrangian for the problem:

$$\mathcal{L}(y, u, \lambda, \xi) = J(y, u) + \langle(\lambda, \xi), G(y, u)\rangle_{(\mathbf{V}\times\mathbf{H}_0^{-1/2}),(\mathbf{V}'\times\mathbf{H}_0^{1/2})}. \tag{19}$$

For ease of notation we do not distinguish here between generic variables $(\lambda, \xi) \in \mathbf{V} \times \mathbf{H}_0^{-1/2}$ and the Lagrange multipliers introduced in Theorem 12.

Lemma 16 *The Lagrange multiplier λ satisfies:*

$$\|\lambda\|_{\mathbf{V}} \leq \theta(y^*)\varrho \|y^* - z_d\|_{\mathbf{H}},$$

where $\theta(y^*) = \frac{1}{\nu - \mathcal{M}(y^*)}$ and ϱ is the constant such that $\|v\|_{\mathbf{H}} \leq \varrho \|v\|_{\mathbf{V}}$, for all $v \in \mathbf{V}$.

Proof. The multiplier λ satisfies

$$\begin{aligned}
\nu \|\lambda\|_{\mathbf{V}}^2 &= (z_d - y^*, \lambda)_{\mathbf{H}} + c(\lambda, \lambda, y^*) \\
&\leq \varrho \|y^* - z_d\|_{\mathbf{H}} \|\lambda\|_{\mathbf{V}} + \mathcal{M}(y^*) \|\lambda\|_{\mathbf{V}}^2,
\end{aligned}$$

which implies

$$(\nu - \mathcal{M}(y^*)) \|\lambda\|_{\mathbf{V}} \leq \varrho \|y^* - z_d\|_{\mathbf{H}}$$

and, hence,

$$\|\lambda\|_{\mathbf{V}} \leq \theta(y^*)\varrho \|y^* - z_d\|_{\mathbf{H}}. \qquad \square$$

Theorem 17 *Let (y^*, u^*) be a stationary point for the constrained optimal control problem. If $2\mathcal{N}\theta(y^*)\varrho \|y^* - z_d\|_{\mathbf{H}} < 1$, then there exists $\kappa > 0$ such that*

$$\langle\mathcal{L}''(y^*, u^*, \lambda, \xi)(w, h), (w, h)\rangle \geq \kappa \|(w, h)\|_{\mathbf{H}\times\mathcal{U}}^2 \tag{20}$$

for all $(w,h) \in \mathbf{H} \times \mathcal{U}$. Thus, (y^*,u^*) is a local optimal solution of the problem.

Proof. We obtain for $(w,h) \in \mathbf{H} \times \mathcal{U}$ that

$$\begin{aligned}\langle \mathcal{L}''(y^*,u^*,\lambda,\xi)(w,h),(w,h)\rangle &= \|w\|_{\mathbf{H}}^2 + \alpha \|h\|^2 + 2c(w,w,\lambda)\\ &\geq \|w\|_{\mathbf{H}}^2 - 2\mathcal{N}\|w\|_{\mathbf{H}}^2 \|\lambda\|_{\mathbf{V}} + \alpha\|h\|^2\\ &= (1-2\mathcal{N}\|\lambda\|_{\mathbf{V}})\|w\|_{\mathbf{H}}^2 + \alpha\|h\|^2\\ &\geq (1-2\mathcal{N}\theta(y^*)\varrho\|y^*-z_d\|_{\mathbf{H}})\|w\|_{\mathbf{H}}^2 + \alpha\|h\|^2\\ &\geq \min(1-2\mathcal{N}\theta(y^*)\varrho\|y^*-z_d\|_{\mathbf{H}},\alpha)\|(w,h)\|_{\mathbf{H}\times\mathcal{U}}^2.\end{aligned}$$

Hence, (20) holds with $\kappa = \min(1-2\mathcal{N}\theta(y^*)\varrho\|y^*-z_d\|_{\mathbf{H}},\alpha)$. Taking in particular (w,h) such that $h \in \mathcal{C}(u^*)$ and $(w,h) \in \ker(G'(y^*,u^*))$, we get (cf. [35]) that (y^*,u^*) is a local minimum for our problem. □

Theorem 18 *The set of stationary points (y^*,u^*) satisfying $2\mathcal{N}\theta(y^*)\varrho\|y^*-z_d\|_{\mathbf{H}} < 1$ consists of a singleton.*

Proof. Let $(y^*,u^*,p^*,\lambda^*,\phi^*,\mu^*,\sigma^*)$ and $(\bar{y},\bar{u},\bar{p},\bar{\lambda},\bar{\phi},\bar{\mu},\bar{\sigma})$ denote two possibly different solutions to the optimality system (14), which satisfy $2\mathcal{N}\theta(y^*)\varrho\|y^*-z_d\|_{\mathbf{H}} < 1$ and $2\mathcal{N}\theta(\bar{y})\varrho\|\bar{y}-z_d\|_{\mathbf{H}} < 1$.

From the seventh equation in (14) we obtain, upon taking the inner product with $\bar{u}-u^*$, that

$$\begin{aligned}\alpha\|\bar{u}-u^*\|^2 = \langle \mathcal{B}^*\left(\tfrac{\partial}{\partial n}(-\bar{y}+y^*-\nu\bar{\lambda}+\nu\lambda^*)+(\bar{\phi}-\phi^*)\vec{n}\right),\bar{u}-u^*\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}}\\ +\langle(\bar{\sigma}-\sigma^*)\vec{n},\bar{u}-u^*\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}} - \langle\bar{\mu}-\mu^*,\bar{u}-u^*\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}}. \quad (21)\end{aligned}$$

Considering that $\bar{u}-u^*$ satisfies the incompressibility condition and applying Green's formula to the first term on the right hand side,

$$\begin{aligned}\alpha\|\bar{u}-u^*\|^2 &= (-\nu\Delta(\bar{\lambda}-\lambda^*)-\Delta(\bar{y}-y^*)+\nabla(\bar{\phi}-\phi^*),\bar{y}-y^*)\\ &\quad -(\nabla(\bar{y}-y^*+\nu\bar{\lambda}-\nu\lambda^*)+(\bar{\phi}-\phi^*)I,\nabla(\bar{y}-y^*)) - \langle\bar{\mu}-\mu^*,\bar{u}-u^*\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}}.\end{aligned}$$

Utilizing the adjoint equation of (14) and taking into account that $((\bar{\phi}-\phi^*)I,\nabla(\bar{y}-y^*)) = \int_\Omega(\bar{\phi}-\phi^*)\ \mathrm{div}\ (\bar{y}-y^*)\,dx = 0$, we get that

$$\begin{aligned}\alpha\|\bar{u}-u^*\|^2 = -\|\bar{y}-y^*\|^2 + c(\bar{y},\bar{\lambda},\bar{y}-y^*) - c(y^*,\lambda^*,\bar{y}-y^*) - c(\bar{y}-y^*,\bar{y},\bar{\lambda})\\ + c(\bar{y}-y^*,y^*,\lambda^*) - \|\nabla(\bar{y}-y^*)\|^2 - \nu(\nabla(\bar{y}-y^*),\nabla(\bar{\lambda}-\lambda^*)) - \langle\bar{\mu}-\mu^*,\bar{u}-u^*\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}}.\end{aligned}$$

Using the primal equation of (14) in variational form it then follows that

$$\alpha \|\bar{u} - u^*\|^2 = - \|\bar{y} - y^*\|_{\mathbf{H}}^2 + c(\bar{y}, \bar{\lambda}, \bar{y} - y^*) - c(y^*, \lambda^*, \bar{y} - y^*) - c(\bar{y} - y^*, \bar{y}, \bar{\lambda}) \\ + c(\bar{y} - y^*, y^*, \lambda^*) - c(\bar{y}, \bar{y}, \lambda^* - \bar{\lambda}) + c(y^*, y^*, \lambda^* - \bar{\lambda}) - \langle \bar{\mu} - \mu^*, \bar{u} - u^* \rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}},$$

which, from the properties of the trilinear form, yields

$$\alpha \|\bar{u} - u^*\|^2 + \|\bar{y} - y^*\|_{\mathbf{H}}^2 + c(\bar{y}_- y^*, \bar{y} - y^*, \bar{\lambda}) + \\ c(\bar{y} - y^*, \bar{y} - y^*, \lambda^*) = -\langle \bar{\mu} - \mu^*, \bar{u} - u^* \rangle_{\mathbf{H}_{00}^{-1/2}, \mathbf{H}_{00}^{1/2}}. \quad (22)$$

Due to the cone properties of μ^* and $\bar{\mu}$ and the definition of $\mathcal{N}$ we obtain that

$$\alpha \|\bar{u} - u^*\|^2 + \|\bar{y} - y^*\|_{\mathbf{H}}^2 - \mathcal{N} \|\bar{y} - y^*\|_{\mathbf{H}}^2 \left\|\bar{\lambda}\right\|_{\mathbf{V}} - \mathcal{N} \|\bar{y} - y^*\|_{\mathbf{H}}^2 \|\lambda^*\|_{\mathbf{V}} \leq 0.$$

From Lemma 16 we have $\|\lambda^*\|_{\mathbf{V}} \leq \theta(y^*)\varrho \|y^* - z_d\|_{\mathbf{H}}$ and $\left\|\bar{\lambda}\right\|_{\mathbf{V}} \leq \theta(\bar{y})\varrho \|\bar{y} - z_d\|_{\mathbf{H}}$. Therefore $1 - \mathcal{N} \left\|\bar{\lambda}\right\|_{\mathbf{V}} - \mathcal{N} \|\lambda^*\|_{\mathbf{V}} > 0$, which implies $\|\bar{u} - u^*\| + \|\bar{y} - y^*\|_{\mathbf{H}} = 0$ and the desired uniqueness follows. □

4 Semi-smooth Newton method for a class of regularized problems

In this section we analyze convergence properties of a semi-smooth Newton method applied to constrained boundary optimal control of the Navier-Stokes equations. Direct application of the method to the infinite dimensional problem is not possible, due to the lack of sufficient regularity of multiplier for the pointwise control constraint in the optimality system (14). Thus, some alternative approach will be utilized in order to have a well posed algorithm.

Our approach is based on a regularization of the original control problem. The idea is to use an appropriate approximation of the multiplier and to apply the semi-smooth Newton method to this transformed problem. Besides proving super-linear convergence of the method for each approximation, convergence of the regularized solutions to the optimal solution has to be shown.

As we saw in Subsection 3.5, the optimality system for the boundary control problem of the Navier-Stokes equations is given by:

$$\begin{cases} -\nu\Delta y^* + (y^*\cdot\nabla)y^* + \nabla p = f \\ \text{div } y = 0 \\ y|_\Gamma = g + \mathcal{B}u^* \\ -\nu\Delta\lambda - (y^*\cdot\nabla)\lambda + (\nabla y^*)^T\lambda + \nabla\phi = (I-\Delta)(z_d - y^*) \\ \text{div } \lambda = 0 \\ \lambda|_\Gamma = 0 \\ \mu = \mathcal{B}^*(\frac{\partial}{\partial n}(-\nu\lambda + z_d - y^*) + \phi\vec{n}) - \alpha u^* + \sigma\vec{n} \quad \text{in } \mathbf{H}_{00}^{-1/2}(\Gamma_1) \\ u^* \leq b \\ \langle \mu, v - b\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}} \leq 0, \text{ for all } v \in \mathbf{H}_{00}^{-1/2}(\Gamma_1), \; v \leq b \\ \langle \mu, u^* - b\rangle_{\mathbf{H}_{00}^{-1/2},\mathbf{H}_{00}^{1/2}} = 0. \end{cases} \tag{23}$$

In view of previous work on optimal control for linear systems with distributed control a natural approach consists in reformulating the last three equations of the system as an operator equation which involves the *max* function [28]. In the present case, however, since μ is not an a.e. defined function, such a procedure does not appear to be possible.

Following an idea introduced for control problems with state constraints (cf. [31]) we approximate the last three equations of the system by

$$\mu_\gamma = \max(0, \gamma(u_\gamma - b)),$$

where $\gamma > 0$. The resulting system is suitable for application of the semi-smooth Newton method and will be shown to converge to the original one as $\gamma \to \infty$.

Let us consider the family of regularized optimal control problems

$$\begin{cases} \min\limits_{(y,u)\in\mathbf{H}\times\mathcal{U}} \; J_\gamma(y,u) = \frac{1}{2}\|y - z_d\|^2_{\mathbf{H}} + \\ \qquad\qquad \frac{\alpha}{2}\|u\|^2 + \frac{1}{2\gamma}\|\max(0,\gamma(u-b))\|^2 \\ \text{subject to} \\ -\nu\Delta y + (y\cdot\nabla)y + \nabla p = f \\ -\text{div } y = 0 \\ \gamma_0 y = g + \mathcal{B}u, \end{cases} \tag{$\mathcal{P}_\gamma$}$$

where $\gamma > 0$.

Theorem 19 *For every $\gamma > 0$ there exists a solution $(y_\gamma, u_\gamma) \in \mathbf{H}\times\mathcal{U}$ to $(\mathcal{P}_\gamma)$. If*

$\mathcal{M}(y_\gamma) < \nu$ then there exist $(\lambda_\gamma, \phi_\gamma, \sigma_\gamma, \mu_\gamma) \in \mathbf{V} \times L_0^2(\Omega) \times \mathbb{R} \times \mathbf{L}^2(\Omega)$ such that

$$\begin{cases} -\nu\Delta y_\gamma + (y_\gamma \cdot \nabla) y_\gamma + \nabla p_\gamma = f \\ div\ y_\gamma = 0 \\ y_\gamma|_\Gamma = g + \mathcal{B}u_\gamma \\ -\nu\Delta\lambda_\gamma - (y_\gamma \cdot \nabla)\lambda_\gamma + (\nabla y_\gamma)^T \lambda_\gamma + \nabla\phi_\gamma = (I - \Delta)(z_d - y_\gamma) \\ div\ \lambda_\gamma = 0 \\ \lambda_\gamma|_\Gamma = 0 \\ \mu_\gamma = \mathcal{B}^*(\frac{\partial}{\partial n}(-\nu\lambda_\gamma + z_d - y_\gamma) + \phi_\gamma \vec{n}) - \alpha u_\gamma + \sigma_\gamma \vec{n} \quad in\ \mathbf{H}_{00}^{-1/2}(\Gamma_1) \\ \mu_\gamma = \max(0, \gamma(u_\gamma - b)) \end{cases} \tag{24}$$

is satisfied in the variational sense.

Proof. Since $J_\gamma(y, u)$ is weakly lower semi-continuous, existence of an optimal solution (y_γ, u_γ) to $(\mathcal{P}_\gamma)$ can be proved by the same arguments as in Theorem 9. By Lemma 10, (y_γ, u_γ) satisfies the regular point condition. Now it can be argued as in the proof of Theorem 14 that there exist $(\lambda_\gamma, \phi_\gamma, \sigma_\gamma, \mu_\gamma) \in \mathbf{V} \times L_0^2(\Omega) \times \mathbb{R} \times \mathbf{L}^2(\Omega)$ such that the optimality system (24) holds. □

The following result on the convergence of (y_γ, u_γ) as $\gamma \to \infty$ will be proved at the end of this section.

Theorem 20 *The family $\{(y_\gamma, u_\gamma)\}_{\gamma>0}$ contains a subsequence converging in $\mathbf{H} \times \mathcal{U}$ and the cluster point of every convergent subsequence is a solution to $(\mathcal{P})$. If*

$$2\mathcal{N}\varrho\ \sup\theta(y^*)\,\|y^* - z_d\|_{\mathbf{H}} < 1, \tag{25}$$

where $\theta(y^*) = \frac{1}{\nu - \mathcal{M}(y^*)}$ and the sup *is taken over all solutions to $(\mathcal{P})$, then the solution to $(\mathcal{P})$ is unique and (y_γ, u_γ) converges in $\mathbf{H} \times \mathcal{U}$ to the solution (y^*, u^*) of $(\mathcal{P})$.*

Based on Theorem 20, convergence of the adjoint variables as $\gamma \to \infty$ can be obtained as well. Assume, for example, that $\lim_{n\to\infty}(y_{\gamma_n}, u_{\gamma_n}) = (\hat{y}, \hat{u})$, and that $\nu > \mathcal{M}(\hat{y})$. Then for n sufficiently large we have $\nu > \mathcal{M}(y_{\gamma_n})$ and the optimality system admits a solution. Let $\delta_y = y_{\gamma_n} - \hat{y}$ and $\delta_\lambda = \lambda_{\gamma_n} - \hat{\lambda}$. Then

$$\begin{cases} -\nu\Delta\delta_\lambda - (\delta_y \cdot \nabla)\hat{\lambda} - (y_{\gamma_n} \cdot \nabla)\delta_\lambda \\ \qquad + (\nabla y_{\gamma_n})^T \delta_\lambda + (\nabla\delta_y)^T \hat{\lambda} + \nabla\delta_\phi = (I - \Delta)(-\delta_y) \\ \text{div } \delta_\lambda = 0 \\ \gamma_0 \delta_\lambda = 0 \end{cases}$$

and hence $\lim_{n\to\infty} \lambda_{\gamma_n} = \hat{\lambda}$ in $\mathbf{V}$.

We turn to the statement of the algorithm of the semi-smooth Newton method or equivalently the primal-dual active set strategy with one inner iteration for nonlinear optimal control problems (cf. [12,32]). The algorithm can be expressed as:

Algorithm

(1) Initialization: choose $(u_0,\ y_0,\ \lambda_0) \in \mathcal{U} \times \mathbf{V} \times \mathbf{H}$ with $\gamma_0 y_0 = g + \mathcal{B}u_0$ and set $n = 1$.

(2) Until a stopping criteria is satisfied, set

$$\mathcal{A}_n = \{x : \gamma(u_{n-1} - b) > 0\}, \qquad \mathcal{I}_n = \Gamma_1 \backslash \mathcal{A}_n.$$

Find the solution $(y_n, p_n, u_n, \lambda_n, \phi_n, \sigma_n)$ of:

$$\begin{aligned}
-\nu\Delta y_n + (y_{n-1}\cdot\nabla)y_n + (y_n\cdot\nabla)y_{n-1} + \nabla p_n &= f + (y_{n-1}\cdot\nabla)y_{n-1}\\
-\text{div}\ y_n &= 0\\
y_n|_\Gamma &= g + \mathcal{B}u_n\\
-\nu\Delta\lambda_n - (y_n\cdot\nabla)\lambda_{n-1} - (y_{n-1}\cdot\nabla)\lambda_n + (\nabla y_{n-1})^T\lambda_n &+ (\nabla y_n)^T\lambda_{n-1}\\
+\nabla\phi_n &= (I-\Delta)(z_d - y_n) - (y_{n-1}\cdot\nabla)\lambda_{n-1} + (\nabla y_{n-1})^T\lambda_{n-1}\\
-\text{div}\ \lambda_n &= 0\\
\lambda_n|_\Gamma &= 0\\
\alpha u_n = \mathcal{B}^*(\frac{\partial}{\partial n}(z_d - y_n - \nu\lambda_n) + \phi_n\vec{n}) + \sigma_n\vec{n} &- \begin{cases}\gamma(u_n - b) & \text{in } \mathcal{A}_n\\ 0 & \text{in } \mathcal{I}_n\end{cases}.
\end{aligned}$$

Set $\mu_n = \begin{cases}\gamma(u_n - b) & \text{in } \mathcal{A}_n\\ 0 & \text{in } \mathcal{I}_n\end{cases}$ and $n = n+1$.

Let us note that the system to be solved in step (2) results from linearization of system (24). It also corresponds to the optimality system of the following optimal control problem:

$$\begin{cases}
\min\limits_{\delta_x \in \mathbf{H}\times\mathcal{U}} \ \frac{1}{2}\langle \mathcal{L}''(x_{n-1}, \lambda_{n-1}, \xi_{n-1})\delta_x, \delta_x\rangle + \\
\qquad\qquad \langle \mathcal{L}'(x_{n-1}, \lambda_{n-1}, \xi_{n-1}), \delta_x\rangle + \frac{1}{2\gamma}\int\limits_{\mathcal{A}_n} |\gamma(u_{n-1} + \delta_u - b)|^2\ d\Gamma\\
\text{subject to}\\
-\nu\Delta\delta_y + (\delta_y\cdot\nabla)y_{n-1} + (y_{n-1}\cdot\nabla)\delta_y + \nabla\delta_p = f\\
\qquad\qquad +\nu\Delta y_{n-1} - (y_{n-1}\cdot\nabla)y_{n-1} - \nabla p_{n-1}\\
-\text{div}\ \delta_y = 0\\
\gamma_0\delta_y - \mathcal{B}\delta_u = -\gamma_0 y_{n-1} + g + \mathcal{B}u_{n-1} \text{ in } \mathbf{H}^{1/2}(\Gamma)
\end{cases} \tag{26}$$

where $x_n = (y_n, u_n)$ and $\delta_x = x_n - x_{n-1}$.

Let us briefly comment on uniqueness of solutions to (26). Assume that $(\mathcal{P}_\gamma)$ satisfies a second order sufficient optimality condition in the sense that for some $\kappa > 0$,

$$\langle \mathcal{L}''(y_\gamma, u_\gamma, \lambda_\gamma, \xi_\gamma)(w,h),(w,h)\rangle \geq \|(w,h)\|^2_{\mathbf{H}\times\mathcal{U}}\,, \tag{27}$$

for all $(w,h) \in \mathbf{H}\times\mathcal{U}$. In view of Theorem 17 this condition holds, for example, if $\|y_\gamma - z_d\|_{\mathbf{H}}$ is sufficiently small. Problem (26) is a quadratic optimization problem with affine constraints. Its Hessian $\bar{H}$ is given by

$$\bar{H}(w,h) = \langle \mathcal{L}''(x_{n-1}, \lambda_{n-1}, \xi_{n-1})(w,h),(w,h)\rangle + \gamma \int_{\mathcal{A}_n} h^2\, d\Gamma,$$

for $(w,h) \in \mathbf{H}\times\mathcal{U}$. Consequently, if (27) holds, then

$$\begin{aligned}
\bar{H}(w,h) &\geq \langle \mathcal{L}''(y_\gamma, u_\gamma, \lambda_\gamma, \xi_\gamma)(w,h),(w,h)\rangle \\
&\qquad + \langle (\mathcal{L}''(x_{n-1}, \lambda_{n-1}, \xi_{n-1}) - \mathcal{L}''(y_\gamma, u_\gamma, \lambda_\gamma, \xi_\gamma))(w,h),(w,h)\rangle \\
&\geq \kappa\,\|(w,h)\|^2_{\mathbf{H}\times\mathcal{U}} - 2\mathcal{N}\,\|w\|_{\mathbf{H}}\,\|\lambda_{n-1} - \lambda_\gamma\|_{\mathbf{V}} \\
&\geq (\kappa - 2\mathcal{N}\,\|\lambda_{n-1} - \lambda_\gamma\|_{\mathbf{V}})\,\|(w,h)\|^2_{\mathbf{H}\times\mathcal{U}}\,,
\end{aligned}$$

and the Hessian to (26) is positive definite if $\|\lambda_{n-1} - \lambda_\gamma\|_{\mathbf{V}}$ is sufficiently small. A sufficient condition for the existence of a Lagrange multiplier $(\delta_\lambda, \delta_\xi)$ to the first and third equality constraint in (26) (resulting in the existence of $\lambda_n = \lambda_{n-1} + \delta_\lambda$ and σ_n in the system above (26)), is given by $\mathcal{M}(y_\gamma) < \nu$ and y_{n-1} sufficiently close to y_γ in $\mathbf{H}$. This is verified as in Theorem 12. The requirements on (y_{n-1}, λ_{n-1}) being sufficiently close to $(y_\gamma, \lambda_\gamma)$ will be justified in Theorem 21 below.

For fixed $\gamma > 0$ and a solution (y_γ, u_γ) to $(\mathcal{P}_\gamma)$ we shall verify super-linear convergence of the semi-smooth Newton method. We shall assume that $\nu > \mathcal{M}(y_\gamma)$ so that existence of a solution to the optimality system (24) is guaranteed. We denote the increments by $\delta_u := u_{n+1} - u_\gamma$, $\delta_y := y_{n+1} - y_\gamma$ and analogously for δ_λ, δ_p, δ_μ and introduce the operators $\mathcal{H} : \mathbf{H}\times\mathbf{H} \to \mathbf{H}'$ and $\widetilde{\mathcal{H}} : \mathbf{H}\times\mathbf{H} \to \mathbf{H}'$ such that $\mathcal{H}(v,w) = (v\cdot\nabla)w$ and $\widetilde{\mathcal{H}}(v,w) = (\nabla v)^T w$. If $v = w$ we use the notation $\mathcal{H}(v) = \mathcal{H}(v,v)$. Using the quadratic nature of the nonlinear form we get:

$$\begin{aligned}
E_1 &:= (y_n - y_\gamma)\cdot\nabla)(y_n - y_\gamma) = \mathcal{H}(y_n) - \mathcal{H}(y_\gamma) - \mathcal{H}'(y_\gamma)(y_n - y_\gamma) \\
&= \frac{1}{2}\mathcal{H}''(y_\gamma)(y_n - y_\gamma)(y_n - y_\gamma) \\
E_2 &:= ((y_n - y_\gamma)\cdot\nabla)(\lambda_n - \lambda_\gamma) \\
&= \mathcal{H}(y_n, \lambda_n) - \mathcal{H}(y_\gamma, \lambda_\gamma) - \mathcal{H}'(y_\gamma, \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma) \\
&= \frac{1}{2}\mathcal{H}''(y_\gamma, \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma) \\
E_3 &:= (\nabla(y_n - y_\gamma))^T(\lambda_n - \lambda_\gamma) \\
&= \widetilde{\mathcal{H}}(y_n, \lambda_n) - \widetilde{\mathcal{H}}(y_\gamma, \lambda_\gamma) - \widetilde{\mathcal{H}}'(y_\gamma, \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma) \\
&= \frac{1}{2}\widetilde{\mathcal{H}}''(y_\gamma, \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma)(y_n - y_\gamma, \lambda_n - \lambda_\gamma).
\end{aligned}$$

Let us also note that, due to the regularity results for the Navier-Stokes and the adjoint equation (see Theorem 8 and the proof of Theorem 15), we obtain that $E_i \in \mathbf{L}^2(\Omega)$, for $i = 1, 2, 3$.

Theorem 21 *If* $\nu > \mathcal{M}(y_\gamma)$, $1 - 2\mathcal{N}\,\|\lambda_\gamma\|_{\mathbf{V}} > 0$ *and* $\|(y_0 - y_\gamma, u_0 - u_\gamma, \lambda_0 - \lambda_\gamma)\|_{\mathbf{H}\times\mathcal{U}\times\mathbf{V}}$ *is sufficiently small, then the sequence* $\{(y_n, u_n, \lambda_n, \mu_n)\}$ *generated by the algorithm converges superlinearly in* $\mathbf{H}\times\mathcal{U}\times\mathbf{V}\times\mathbf{L}^2(\Omega)$ *to* $(y_\gamma, u_\gamma, \lambda_\gamma, \mu_\gamma)$.

Proof. Let $\delta > 0$ denote the positive constant which describes the smallness condition for $\|(y_0 - y_\gamma, u_0 - u_\gamma, \lambda_0 - \lambda_\gamma)\|_{\mathbf{H}\times\mathcal{U}\times\mathbf{V}}$. At first δ is chosen such that

$$\nu - \mathcal{M}(y) \geq \frac{1}{2}(\nu - \mathcal{M}(y_\gamma)) > 0 \qquad \text{and} \qquad 1 - 2\mathcal{N}\,\|\lambda\|_{\mathbf{V}} \geq \frac{1}{2} - \mathcal{N}\,\|\lambda_\gamma\|_{\mathbf{V}} > 0 \quad (28)$$

for all (y, λ) with $\|y - y_\gamma\|_{\mathbf{H}} < \delta$ and $\|\lambda - \lambda_\gamma\|_{\mathbf{V}} < \delta$. Further below the value of δ will be decreased. We proceed by induction and assume that $\|y_i - y_\gamma\|_{\mathbf{H}} < \delta$, $\|u_i - u_\gamma\|_{\mathcal{U}} < \delta$ and $\|\lambda_i - \lambda_\gamma\|_{\mathbf{V}} < \delta$ for all $i = 0, ..., n$. In the induction step we show that these inequalities also hold for $i = n + 1$, as well as provide the superlinear convergence estimate.

Considering the systems satisfied by the regularized solution $(y_\gamma, p_\gamma, u_\gamma, \lambda_\gamma, \xi_\gamma, \sigma_\gamma, \mu_\gamma)$ and the iterate $(y_n, p_n, u_n, \lambda_n, \xi_n, \sigma_n, \mu_n)$, it can be verified that the system

$$\begin{cases} -\nu\Delta\delta_y + (y_n\cdot\nabla)\delta_y + (\delta_y\cdot\nabla)y_n + \nabla\delta_p = E_1 \\ -\text{div }\delta_y = 0 \\ \delta_y|_\Gamma = \mathcal{B}\delta_u \\ \int_\Gamma \delta_u\cdot\vec{n}\; d\Gamma = 0 \\ -\nu\Delta\delta_\lambda - (y_n\cdot\nabla)\delta\lambda - (\delta_y\cdot\nabla)\lambda_n + (\nabla y_n)^T\delta\lambda \\ \qquad\qquad +(\nabla\delta_y)^T\lambda_n + \nabla\delta_\phi = E_3 - E_2 + \Delta\delta_y - \delta_y \\ \text{div }\delta_\lambda = 0 \\ \delta_\lambda|_\Gamma = 0 \\ \alpha\delta_u + \mathcal{B}^*\left(\frac{\partial}{\partial n}(\nu\delta_\lambda + \delta_y) - \delta_\phi\vec{n}\right) - \delta_\sigma\vec{n} = -\delta_\mu \\ \delta_\mu = \gamma G(\gamma(u_n - b))\delta_u + R, \end{cases} \quad (29)$$

with

$$R = \max(0, \gamma(u_\gamma + (u_n - u_\gamma) - b)) - \max(0, \gamma(u_\gamma - b)) \\ - \gamma G(\gamma(u_\gamma + (u_n - u_\gamma) - b))(u_n - u_\gamma)$$

and

$$G(g)(x) = \begin{cases} 0 & \text{if } g(x) \leq 0 \\ 1 & \text{if } g(x) > 0, \end{cases}$$

holds in the variational sense.

From the Newton differentiability of the *max* function (cf. [28]) we know that for each $g \in L^p(\Gamma_1)$,

$$\|\max(0, g+h) - \max(0, g) - G(g+h)h\|_{L^2} = o(\|h\|_{L^p}). \tag{30}$$

From the equation for δ_λ in (29) we obtain

$$\begin{aligned}\nu \|\nabla \delta_\lambda\|^2 - c(\delta_y, \lambda_n, \delta_\lambda) + c(\delta_\lambda, \delta_y, \lambda_n)& \\ + c(\delta_\lambda, y_n, \delta_\lambda) = (E_3 - E_2, \delta_\lambda) &- (\nabla \delta_y, \nabla \delta_\lambda) - (\delta_y, \delta_\lambda) \end{aligned} \tag{31}$$

and consequently from (28) and Lemma

$$\begin{aligned}\frac{1}{2}(\nu - \mathcal{M}(y_\gamma)) \|\nabla \delta_\lambda\|^2 \leq 2\mathcal{N} \|\delta_y\|_{\mathbf{H}} \|\delta_\lambda\|_{\mathbf{V}} \|\lambda_n\|_{\mathbf{V}}& \\ + C_1 \|y_n - y_\gamma\|_{\mathbf{H}} \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}} \|\delta_\lambda\|_{\mathbf{V}} &- (\delta_y, \delta_\lambda)_{\mathbf{H}}.\end{aligned}$$

Here and below C_i denote generic constants independent of n. This estimate implies the existence of a constant C_2, such that

$$\|\delta_\lambda\|_{\mathbf{V}} \leq C_2(\|\delta_y\|_{\mathbf{H}} + \|y_n - y_\gamma\|^2_{\mathbf{H}} + \|\lambda_n - \lambda_\gamma\|^2_{\mathbf{V}}). \tag{32}$$

From the last two equations in (29) we obtain, after taken the inner product with δ_u,

$$\begin{aligned}-(R, \delta_u) = \alpha \|\delta_u\|^2 + \langle \frac{\partial}{\partial n}(\nu \delta_\lambda + \delta_y) - \delta_\phi \vec{n}, \delta_y \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}}& \\ - \delta_\sigma(\delta_u, \vec{n}) + \gamma(G(\gamma(u_n - b))\delta_u, \delta_u)&.\end{aligned} \tag{33}$$

Applying Green's formula to the second term on the right hand side we obtain

$$\begin{aligned}\langle \frac{\partial}{\partial n}(\nu \delta_\lambda + \delta_y) - \delta_\phi \vec{n}, \delta_y \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} = (\nu \Delta \delta_\lambda + \Delta \delta_y - \nabla \delta_\phi, \delta_y)& \\ + (\nabla(\nu \delta_\lambda + \delta_y) - \delta_\phi I, \nabla \delta_y)&,\end{aligned} \tag{34}$$

which, using the adjoint equations in (29), yields

$$\begin{aligned}\langle \frac{\partial}{\partial n}(\nu \delta_\lambda + \delta_y) - \delta_\phi \vec{n}, \delta_y \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} = \|\delta_y\|^2 - (E_3 - E_2, \delta_y) - c(y_n, \delta_\lambda, \delta_y)& \\ - c(\delta_y, \lambda_n, \delta_y) + c(\delta_y, y_n, \delta_\lambda) + c(\delta_y, \delta_y, \lambda_n) + \nu(\nabla \delta_y, \nabla \delta_\lambda) + \|\nabla \delta_y\|^2 - (\delta_\phi I, \nabla \delta_y)&.\end{aligned}$$

Utilizing the primal equations of (29) in variational form and taking into account that $(\delta_\phi I, \nabla \delta_y) = \int_\Omega \delta_\phi \operatorname{div} \delta_y \, dx = 0$, we obtain

$$\begin{aligned}\langle \frac{\partial}{\partial n}(\nu \delta_\lambda + \delta_y) - \delta_\phi \vec{n}, \delta_y \rangle_{\mathbf{H}^{-1/2}, \mathbf{H}^{1/2}} = \|\delta_y\|^2_{\mathbf{H}} - (E_3 - E_2, \delta_y) - c(y_n, \delta_\lambda, \delta_y)& \\ - c(\delta_y, \lambda_n, \delta_y) + c(\delta_y, y_n, \delta_\lambda) + c(\delta_y, \delta_y, \lambda_n) + (E_1, \delta_\lambda) - c(y_n, \delta_y, \delta_\lambda) - c(\delta_y, y_n, \delta_\lambda)&.\end{aligned}$$

Considering that δ_u satisfies the incompressibility condition $\int_\Gamma \delta_u \cdot \vec{n}\, d\Gamma = 0$ and using the properties of the trilinear form, we get

$$-(R, \delta_u) = \alpha \|\delta_u\|^2 + \|\delta_y\|_{\mathbf{H}}^2 - (E_3 - E_2, \delta_y) - 2c(\delta_y, \lambda_n, \delta_y) \\ + (E_1, \delta_\lambda) + \gamma(G(\gamma(u_n - b))\delta_u, \delta_u).$$

From (28) we have

$$\alpha \|\delta_u\|^2 + (\frac{1}{2} - \mathcal{N} \|\lambda_\gamma\|_{\mathbf{V}}) \|\delta_y\|_{\mathbf{H}}^2 \\ \leq \|R\| \|\delta_u\| + C_1(\|y_n - y_\gamma\|_{\mathbf{H}}^2 \|\delta_\lambda\|_{\mathbf{V}} + \|y_n - y_\gamma\|_{\mathbf{H}} \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}} \|\delta_y\|_{\mathbf{H}}).$$

Using (30) with $h = \gamma(u_n - u_\gamma)$ and $g = \gamma(u_\gamma - b)$, we obtain $\|R\| = o(\|u_n - u_\gamma\|_{\mathbf{L}^p})$ and, by the injection $\mathbf{H}^{1/2}(\Gamma) \hookrightarrow \mathbf{L}^p(\Gamma)$, further $\|R\| = o(\|u_n - u_\gamma\|_{\mathcal{U}})$. From (32) we deduce

$$\alpha \|\delta_u\|^2 + (\frac{1}{2} - \mathcal{N} \|\lambda_\gamma\|_{\mathbf{V}}) \|\delta_y\|_{\mathbf{H}}^2 \leq o(\|u_n - u_\gamma\|_{\mathcal{U}}) \|\delta_u\| \\ + C_3(\|y_n - y_\gamma\|_{\mathbf{H}}^2 \|\delta_y\|_{\mathbf{H}} + \|y_n - y_\gamma\|_{\mathbf{H}}^4 + \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}}^4 + \|y_n - y_\gamma\|_{\mathbf{H}} \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}} \|\delta_y\|_{\mathbf{H}}).$$

From this estimate and the fact that $\gamma_0 \delta_y = \mathcal{B}\delta_u$ we deduce the existence of a constant C independent of n such that

$$\|(\delta_y, \delta_u)\|_{\mathbf{H}\times\mathcal{U}} \leq C(\|y_n - y_\gamma\|_{\mathbf{H}}^2 + \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}}^2) + o(\|u_n - u_\gamma\|_{\mathcal{U}}).$$

Referring to (32) once again and to the last equation of (29) we obtain for an appropriately modified constant C

$$\|(\delta_y, \delta_u, \delta_\lambda, \delta_\mu)\|_{\mathbf{H}\times\mathcal{U}\times\mathbf{V}\times\mathbf{L}^2(\Omega)} \leq C(\|y_n - y_\gamma\|_{\mathbf{H}}^2 + \|\lambda_n - \lambda_\gamma\|_{\mathbf{V}}^2) + o(\|u_n - u_\gamma\|_{\mathcal{U}}). \quad (35)$$

This implies, possibly after reducing δ again, that $\|y_{n+1} - y_\gamma\|_{\mathbf{H}} < \delta$, $\|u_{n+1} - u_\gamma\|_{\mathcal{U}} < \delta$, $\|\lambda_{n+1} - \lambda_\gamma\|_{\mathbf{V}} < \delta$, as well as superlinear convergence. □

Proof of Theorem 20. Let (y^*, u^*) denote a solution to $(\mathcal{P})$. Since $J_\gamma(y_\gamma, u_\gamma) \leq J_\gamma(y^*, u^*) = J(y^*, u^*)$, the family $\{y_\gamma\}_{\gamma>0}$ is bounded in $\mathbf{H}$, and by the trace theorem $\{u_\gamma\}_{\gamma>0}$ is bounded in $\mathbf{H}_{00}^{1/2}(\Gamma_1)$. Consequently there exist subsequences of $\{y_\gamma\}_{\gamma>0}$ and $\{u_\gamma\}_{\gamma>0}$, denoted by the same symbols, and $(\hat{y}, \hat{u}) \in \mathbf{H} \times \mathbf{H}_{00}^{1/2}(\Gamma_1)$ such that (y_γ, u_γ) converges weakly in $\mathbf{H} \times \mathbf{H}_{00}^{1/2}(\Gamma_1)$ to $(\hat{y}, \hat{u})$. Moreover $\frac{1}{\gamma} \|\max(0, \gamma(u_\gamma - b))\|^2$ is bounded and hence $\lim_{\gamma\to\infty} \|\max(0, u_\gamma - b)\| = 0$. By Fatou's Lemma, this implies that $\max(0, \hat{u} - b) = 0$ and hence $\hat{u} \leq b$, i.e. $\hat{u}$ is admissible. We also have

$$J(\hat{y}, \hat{u}) \leq \underline{\lim}\ J(y_\gamma, u_\gamma) \leq \underline{\lim}\ J_\gamma(y_\gamma, u_\gamma) \leq J(y^*, u^*)$$

and hence $(\hat{y}, \hat{u})$ is a solution to $(\mathcal{P})$. Moreover

$$\lim_{\gamma\to\infty} \|y_\gamma - z_d\|_{\mathbf{H}}^2 + \alpha \|u_\gamma\|_{\mathbf{L}^2(\Gamma_1)}^2 = \|\hat{y} - z_d\|_{\mathbf{H}}^2 + \alpha \|\hat{u}\|_{\mathbf{L}^2(\Gamma_1)}^2,$$

and therefore, utilizing the trace theorem, $y_\gamma \to \hat{y}$, $u_\gamma \to \hat{u}$ strongly in $\mathbf{H} \times \mathcal{U}$. If (25) holds, then by Theorem 18 the solution to $(\mathcal{P})$ is unique and the whole family (y_γ, u_γ) converges strongly in $\mathbf{H} \times \mathcal{U}$ to (y^*, u^*), the solution of $(\mathcal{P})$. □

5 Numerical results

In this section we present some numerical tests, which illustrate the behaviour of the semi-smooth Newton method applied to constrained boundary optimal control of the Navier-Stokes equations.

As domain we use the channel $(0,1) \times (0,0.5)$ and set a step by removing the rectangular sector $(0.5,1) \times (0,0.25)$ from the domain. The fluid flows from left to right and has a parabolic inflow boundary condition with maximum value equal to one. For the outflow boundary condition we use the so called "do nothing" condition (cf. [39]), which was shown to be appropriate in channel simulations. For the rest of the walls a homogeneous Dirichlet condition is imposed, which is of "no slip" type in the sectors where the control does not act. This problem is referred to as "forward facing step flow".

The domain was discretized using a homogeneous staggered grid, with discretization step h, combined with a finite differences scheme. For the solution of the nonlinearity we apply a Newton method. In order to avoid numerical instabilities and to obtain appropriate results for high Reynolds numbers, a first order upwinding scheme is used. For the solution of the linear system in each Newton step we use MATLAB's sparse solver.

The simulation of the fluid is depicted in Figure 1 and Figure 2 for Reynolds numbers 1000 and 1500, respectively. It is clear that as Re increases, the recirculation bubbles become bigger and stronger. In fact, a desirable effect of a control would be to reduce this recirculation effect in order to avoid possible flow separation.

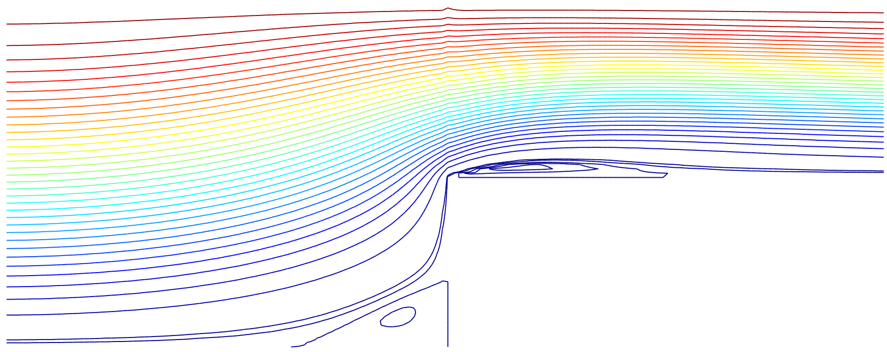

Fig. 1. Plot of the streamlines for the forward facing step flow with $Re = 1000$.

We consider two cases of Dirichlet boundary control. The control will act on Γ_1, which is part of the boundary corresponding to the lower wall after the step, between the values 0.625 and 0.75. In the first case, we allow the control to act as the normal component of the

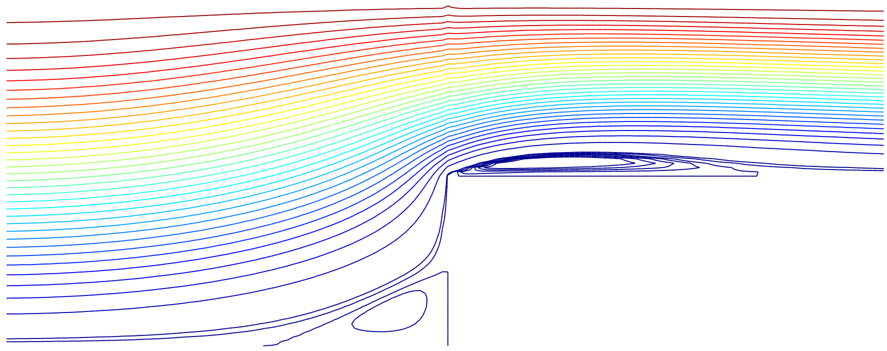

Fig. 2. Plot of the streamlines for the forward facing step flow with $Re = 1500$.

velocity at the boundary, while the tangential component is set zero. This case corresponds to normal suction and blowing of fluid along the prescribed wall. In the second case only the tangential component is used as control, while the normal component remains zero. This corresponds to the case where another fluid or a moving band acts on the other side of the domain, with the prescribed condition at the wall.

The target of the control is to drive the fluid to an almost linear behavior, specified by the Navier-Stokes flow, with Reynolds number 1, in the channel.

The cost functional used for the tests contains a weight for the gradient part of the norm, i.e.:

$$J(y,u) = \frac{1}{2}\|y - z_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|\nabla(y - z_d)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{\mathbf{L}^2(\Gamma_1)},$$

where $\beta > 0$ and z_d is the state of the Navier Stokes equation with $Re = 1$.

For the computation of the first two examples we used the regularized semi-smooth Newton method. The method stops if $\|\delta_x\|_{L^2} < \varepsilon = 10^{-4}$. For both examples, the method requires one additional iteration if $\varepsilon = 10^{-7}$.

5.1 *Example 1*

In this first case, we apply a constrained tangential optimal control to drive the stationary Navier-Stokes flow to the desired state. The parameter values used are: $Re = 1000$, $b = 0.425$, $\beta = 10^{-4}$ and $\alpha = 0.01$. The optimal control and its multiplier are depicted in Figure 3, where, with dotted line, the unconstrained optimal control is also shown. From these graphics, the satisfaction of the complementarity condition can be verified by inspection. Figure 4 depicts the streamlines of the controlled state and Figure 5 a zoomed view of the velocity vector plot at the bubble sector.

Intuitively, one can imagine that an appropriate control action should contravene the recirculation effect of the bubble by imposing a horizontal velocity in the opposite direction of the recirculation. The numerical results confirm this conjecture. In controlled flow the bubble center and its concentration are moved to that part of the domain where the

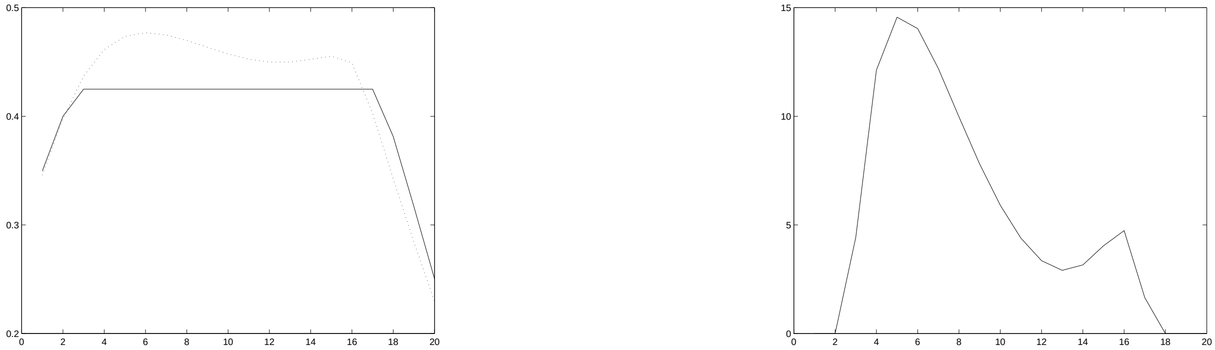

Fig. 3. Example 1: horizontal optimal control and its multiplier.

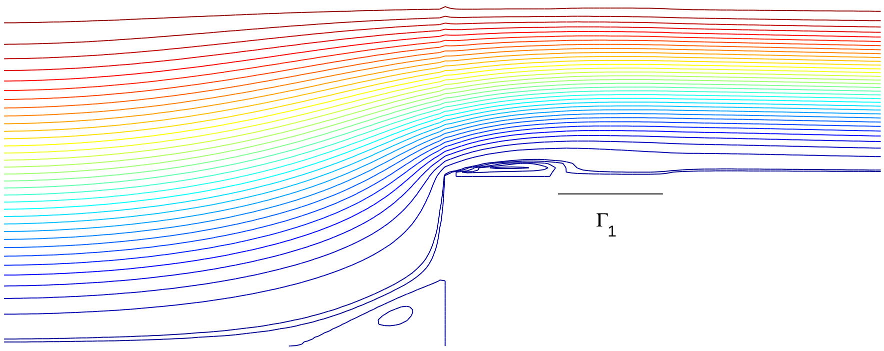

Fig. 4. Example 1: plot of the streamlines of the final controlled state.

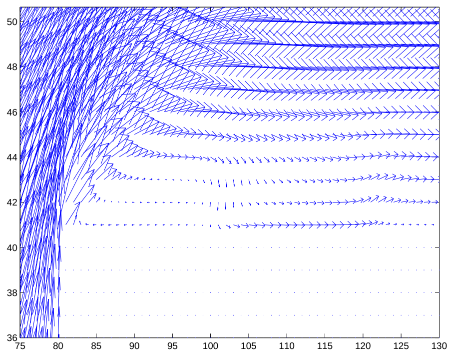

Fig. 5. Example 1: final controlled state: zoom of the vector field.

control has no influence. In a neighborhood of the control, the recirculation disappears.

From the plot of the controlled state it can be seen that the action of the control provokes a significant change in the behaviour of the fluid. Since such changes are penalized by the second term of the cost functional, the question about what to expect as β increases arises.

For larger values of β the design objective in the sense of reducing the recirculation region is less successful. This can be explained by observing that the streamlines in Figure 4 are quite concentrated in the remaining bubble in front of the controlled boundary and that

Table 1
Example 1, h=0.00625, $\varepsilon = 10^{-4}$.

γ	10	10^2	10^4	10^8	10^{10}
Iter.	6	6	8	8	8
$\lvert \mathcal{A} \rvert$	14	14	15	15	15

Table 2
Example 1, h=0.005, $\varepsilon = 10^{-7}$.

Iteration	$\lvert \mathcal{A}_n \rvert$	$\lvert \mathcal{I}_n \rvert$	$J(y,u)$	$\lVert u_n - u_{n-1} \rVert$	$\frac{\lVert u_n - u_{n-1} \rVert}{\lVert u_{n-1} - u_{n-2} \rVert}$
1	25	0	0.00844382	-	-
2	0	25	0.00738098	0.3399	-
3	19	6	0.00722746	0.1969	0.5792
4	18	7	0.00721617	0.0198	0.1005
5	18	7	0.00721604	0.00392	0.1665
6	18	7	0.00721604	$2.36 \cdot 10^{-6}$	$6.98 \cdot 10^{-4}$
7	18	7	0.00721604	$1.15 \cdot 10^{-12}$	$5.03 \cdot 10^{-7}$

as a consequence the gradient norm is large in that region. If the norm does not allow for this behavior the bubble remains larger.

Table 1 shows the iteration number of the primal-dual method for increasing γ values. In Table 2 the evolution of the method and the cost functional are depicted for the fixed γ value 10^{10}. The data were obtained with mesh size $h = 0.005$.

5.2 Example 2

In this example we use normal control action along the prescribed wall section. The unilateral constraint can be interpreted as a limited pointwise suction or injection capacity, depending on the kind of inequality we have. For this case, the suction of fluid is restricted pointwise by imposing a constant lower bound on the control. The parameters take the following values: $Re = 800$, $b = -0.18$, $\beta = 10^{-4}$ and $\alpha = 0.01$. Figure 6 shows, with solid lines, the optimal control and its multiplier and, with dotted lines, the unconstrained optimal control. The streamlines of the final state are given in Figure 7.

The action of the control consists in sucking fluid in order to eliminate the recirculation within the control sector. A displacement of the bubble center takes place and the

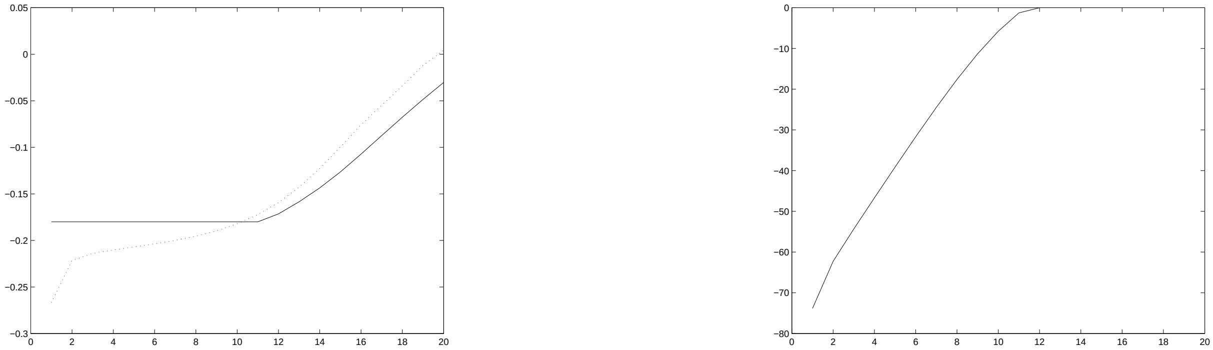

Fig. 6. Example 2: vertical optimal control and its multiplier.

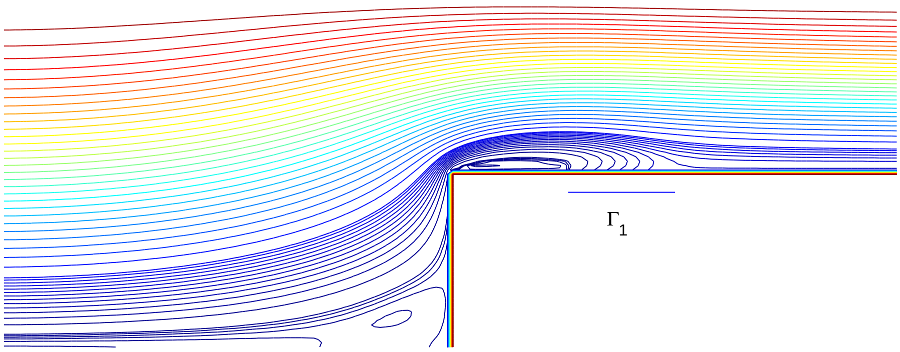

Fig. 7. Example 2: streamlines of the final controlled state.

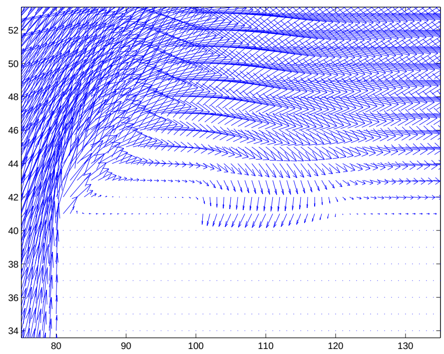

Fig. 8. Example 2: final controlled state: zoom of the vector field.

recirculation occurs earlier. Figure 8 shows a zoom of the bubble, which illustrates the control effect. The complementarity condition can be confirmed by inspection from Figure 6. Since the constraint consist in a lower bound, the multiplier takes negative values.

It can be also seen from the plots that, differently from tangential control, in the case of normal controls the qualitative properties of the controls and of the fluid do not change significantly with β. Figure 9 shows the state reached by taking $\beta = 0.1$. In this case the constraint remains active on the whole control domain.

Table 3 shows the evolution of the primal-dual method and the cost functional for $\beta =$

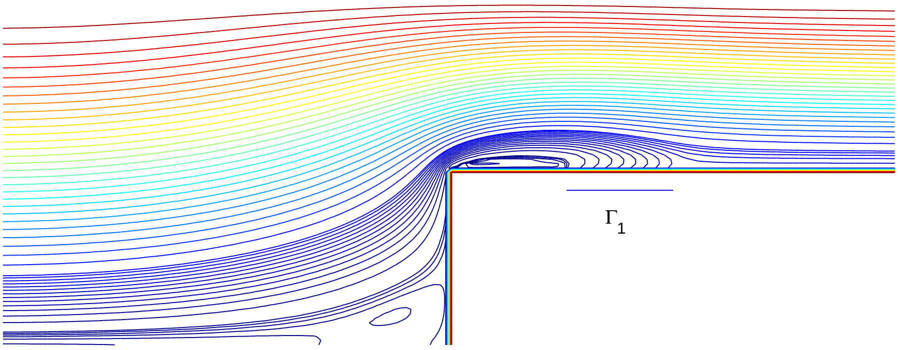

Fig. 9. Example 2: streamlines of the final controlled state, $\beta = 0.1$.

Table 3
Example 2, h=0.005, $\varepsilon = 10^{-4}$.

Iteration	$\mid \mathcal{A}_n \mid$	$\mid \mathcal{I}_n \mid$	$J(y,u)$	$\|u_n - u_{n-1}\|$	$\frac{\|u_n-u_{n-1}\|}{\|u_{n-1}-u_{n-2}\|}$
1	20	0	0.00612187	-	-
2	0	20	0.00570537	0.444	-
3	6	14	0.00556574	0.2181	0.4908
4	10	10	0.00554262	0.0812	0.3721
5	11	9	0.00554544	0.0013	0.0157
6	11	9	0.00554545	$3.7 \cdot 10^{-6}$	0.0029
7	11	9	0.00554545	$8.7 \cdot 10^{-12}$	$2.35 \cdot 10^{-6}$

Table 4
Example 1, h=0.00625, $\varepsilon = 10^{-4}$.

α	Iterations	$\|\mathcal{A}\|$
0.1	6	3
0.01	7	11
0.001	7	12
0.0001	7	12
0.00001	7	12

10^{-4}, $\gamma = 10^{10}$ and discretization step $h = 0.005$. The behavior of the method as α decreases is given in Table 4 and, as expected, it can be confirmed that with reduction of the weight, the size of the active set becomes larger.

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